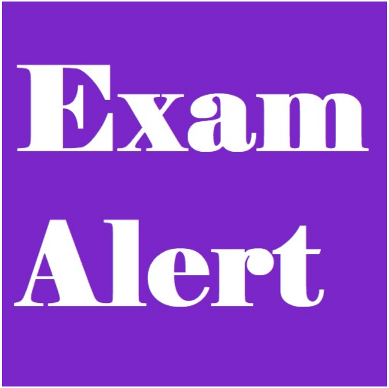
MATH 223: Multivariable Calculus

Class 18: April 1, 2022

Notes on Assignment 17
Assignment 18
Extreme Values (Last Time)



Monday, April 11

Alternative Solution to Problem A

Let $F : \mathbb{R}^3 \to \mathbb{R}^1 : F(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2 - 6$

and

$$g : \mathbb{R}^2 \to \mathbb{R}^3 : g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}$$

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so the derivative $F'(g(\mathbf{x})$ is also identically 0.The Chain Rule gives us

$$\left(\frac{x}{2}, \frac{2y}{9}, 2z \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix} = (0, 0)$$

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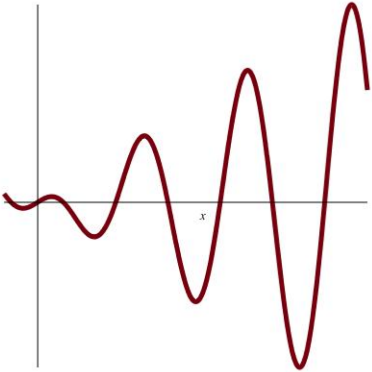
$$\text{Evaluate at } x = 2, y = 3, z = -2:$$

$$1 - 4\, f_x(2,3) = 0 \text{ and } \frac{2}{3} - 4\, f_y(2,3) = 0$$

so

$$f_x(2,3) = \frac{1}{4},\; f_y(2,3) = \frac{1}{6}$$

Today:
Maxima and Minima of Real-Valued Functions

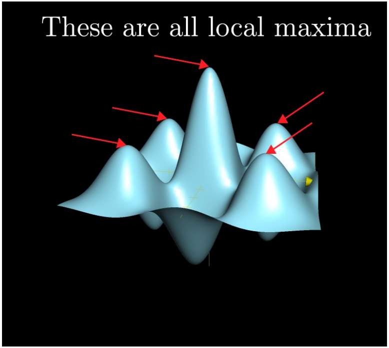


Let D be a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^1$ be a real-valued function with $\vec{x_o}$ a point in D.

Definition: f has an **absolute maximum** at $\vec{x_o}$ if $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in D.

Note: $\geq$ makes sense because we are comparing real numbers.

f has a **relative maximum** at $\vec{x_o}$ if there is a neighborhood N around $\vec{x_o}$ such that $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in N.



Theorem: Let $\vec{x_o}$ be an **interior** point of D. If f is differentiable at $\vec{x_o}$ and f has a relative maximum or minimum at $\vec{x_o}$, then $f'(\vec{x_o}) = \nabla(\vec{x_o}) = \vec{0}$.

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then $f'(\vec{x_o}) = \nabla(\vec{x_o}) = \vec{0}$.

Proof: Suppose f has a relative maximum at $\vec{x_o}$
Let $\vec{u}$ be any unit vector in $\mathbb{R}^n$.

$$\text{Then } \frac{\partial f}{\partial \vec{u}} = \lim_{t \to 0} \frac{f(\vec{x_0} + t\vec{u}) - f(\vec{x_0})}{t}$$

(a) Take $\lim_{t \to 0^+} : \frac{-}{+} \leq 0$

thus $\frac{\partial f}{\partial \vec{u}} = 0$ for all $\vec{u}$

(b) Take $\lim_{t \to 0^-} : \frac{-}{-} \geq 0$

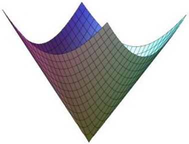
Taking $\vec{u}$ to be unit vectors gives $\nabla f(\vec{x_0}) = 0$

Theorem: f differentiable at relative extrema implies gradient is 0.

The Theorem Has Its Limitations:

(1) **The function can have an extreme value at a point where it is not differentiable.**

Example: $f(x, y) = \sqrt{x^2 + y^2}$ has minimum at (0,0) but is not differentiable there.



$$f_x(x, y) = \frac{x}{\sqrt{x^2+y^2}}, f_y(x, y) = \frac{y}{\sqrt{x^2+y^2}},$$

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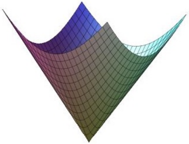
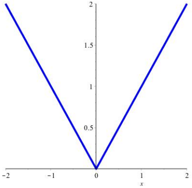


Analogue in Calculus I:

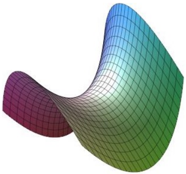
$f(x) = \sqrt{x^2} = |x|$

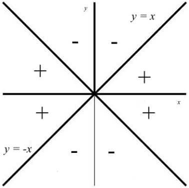


Theorem: f differentiable at relative extrema implies gradient is 0.

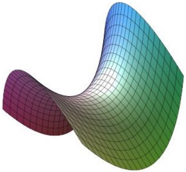
The Theorem Has Its Limitations:

(2) We can have $\nabla f(\vec{x_0}) = 0$ but no extreme point at $\vec{x_0}$





$$\nabla f(x, y) = (2x, 2y)$$



There is a Maximum is one direction and a Minimum in another

Saddle Point

Quiz:
Name a Famous Commercial Food Product That Exhibits a Saddle Point


Pringles
the Original
NET WT/PESO NETO 2.36 OZ

Definition: A point $\vec{x_0}$ in the domain of f is a **Critical Point** of f if

(a) $\nabla f(\vec{x_0}) = \vec{0}$

or

(b) ∇f does not exist at $\vec{x_0}$.

Extreme Values Can Occur at Critical Points or Points on the Boundary

Example: Temperature Distribution on disk of radius 1 centered at origin is $T(x, y) = 2x^2 + 4y^2 + 2x + 1$.

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Note $T(-\frac{1}{2}, 0) = 2(\frac{1}{4}) + 4(0^2) + 2(-\frac{1}{2}) + 1 = \frac{1}{2}$, and $T(0,0) = 1$.

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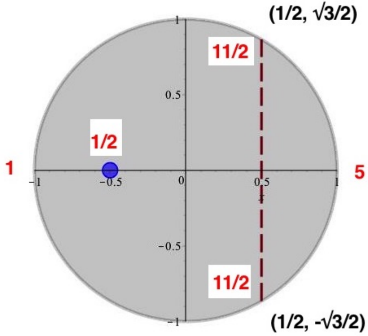
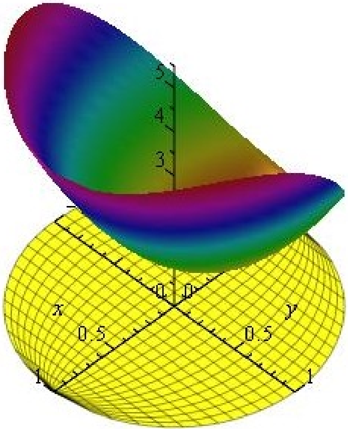


red numbers are values of the function


5
4
3
0
0
x
y
0.5
0.5
1
1

Parametrize Boundary

$x = \cos t, y = \sin t$ for $0 \le t \le 2\pi$

$$\begin{aligned} T(x,y) &= 2x^2 + 4y^2 + 2x + 1 \\ &= 2\cos^2 t + 4\sin^2 t + 2\cos t + 1 \\ &= 2\cos^2 t + 2\sin^2 t + 2\sin^2 t + 2\cos t + 1 \\ &= 2 + 2\sin^2 t + 2\cos t + 1 = 2\sin^2 t + 2\cos t + 3 \\ &= H(t) \end{aligned}$$

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&= 2\cos^2 t + 2\sin^2 t + 2\sin^2 t + 2\cos t + 1 \\
&= 2 + 2\sin^2 t + 2\cos t + 1 = 2\sin^2 t + 2\cos t + 3 \\
&= H(t)
\end{aligned}
$$

$H(0) = 2\cdot 1 + 2\cdot 0 + 3 = 5, H(\pi) = 2\cdot 1 + 2\cdot -1 + 3 = 1$

Now $H'(t) = 4\sin t\cos t - 2\sin t = 2\sin t(2\cos t - 1)$ so

$H'(t) = 0$ at $\sin t = 0$ or $\cos t = \frac{1}{2}$

The first condition gives $t = 0, t = \pi$, the second occurs when $t = \frac{\pi}{3}$.

Next Time:

Solving Constrained Optimization Problems Using Lagrange Multipliers