CLASS 13

HANDOUTS

NOTES ON ASSIGNMENT 11

Assignment 12

THEOREM: If f is differentiable at $\vec{a}$,
Then $f_{\vec{v}}(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$

DIRECTIONAL DERIVATIVE of f at $\vec{a}$ IN direction $\vec{v}$ is $f_{\vec{u}}(\vec{a})$ where $\vec{u} = \vec{v}/|\vec{v}|$

for differentiable fns:

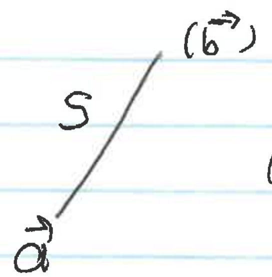
$$f_{\vec{u}}(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$$
$$= |\nabla f(\vec{a})|\,|\vec{u}| \cos\theta$$
$$= |\nabla f(\vec{a})| \cos\theta$$

which is maximized when $\cos\theta = 1 \Rightarrow \theta = 0$

Maximize rate of change by moving in direction of gradient.

MEAN VALUE THEOREM: $f: \mathbb{R}^n \to \mathbb{R}^1$

ASSUME f IS differentiable on each point of S, a line segment between $\vec{a}$ and $\vec{b}$.



Then there IS AT LEAST ONE $\vec{c}$ ON S WITH

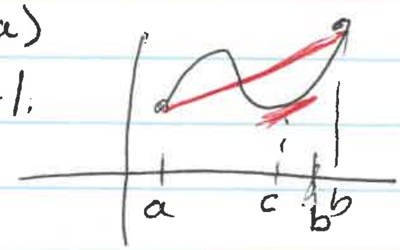
$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$$

Recall CLASSIC MVT from Single VARIABLE Calculus

$f: \mathbb{R}^1 \to \mathbb{R}^1$ differentiable on $[a,b]$

$\Rightarrow \quad f(b) - f(c) = f'(c)\,(b-a)$

for some c inside interval.



ONE IMPORTANT CONSEQUENCE

If $f'(x) = g'(x)$ for all x in $[a,b]$

Then $f(x) = g(x) + C$ for all x and some constant C

proof (using CLASSIC MVT):

Let $H(x) = f(x) - g(x)$

Then $H'(x) = f'(x) - g'(x) = 0$ for all x

CLAIM: H IS constant on $[a,b]$

Let $x_1 < x_2$ be any 2 points in interval.

Then $H(x_2) - H(x_1) = H'(c)\,(x_2 - x_1) = 0$

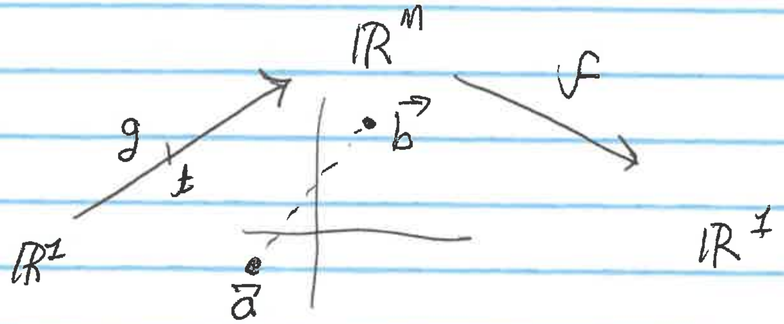
so $H(x_2) = H(x_1)$ $\square$

THE SAME ARGUMENT SHOWS

$$\nabla f \equiv \nabla g \Rightarrow f(\vec{x}) = g(\vec{x}) + C$$

for some constant C

PROOF OF GENERALIZED MVT



Define a new function $\vec{g}: [0, 1] \to \mathbb{R}^n$
by $\vec{g}(t) = \vec{a} + t(\vec{b} - \vec{a})$
NOTE $\vec{g}(0) = \vec{a}$ and $\vec{g}(1) = \vec{b}$
$\vec{g}(t)$ lies on S and $g'(t) = \vec{b} - \vec{a}$

CONSIDER THE COMPOSITION.
$H(t) = f(g(t))$ which maps $\underset{[0,1]}{\mathbb{R}^1} \to \mathbb{R}^1$

BY CLASSIC MVT
$$H(1) - H(0) = H'(t_c) \cdot [1 - 0] = H'(t_c)$$
But $H(1) = f(g(1)) = f(\vec{b})$
and $H(0) = f(g(0)) = f(\vec{a})$

WHAT IS $H'(t)$? CHAIN RULE
$$\underbrace{f'(g(t))}_{\nabla f(g(t))} \underbrace{g'(t)}_{(\vec{b} - \vec{a})}$$

so $H'(t_c) = \nabla f(g(t_c)) \cdot \vec{b} - \vec{a})$
Let $\vec{c} = g(t_c)$
THUS $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$

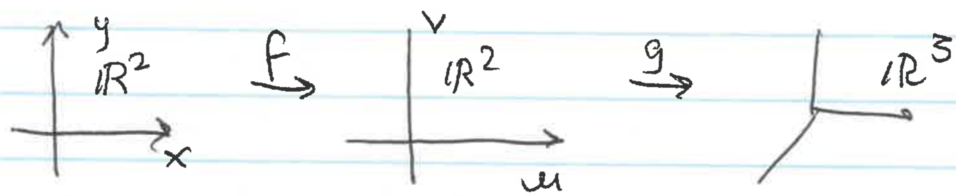
CHAIN RULE

$$\begin{tikzcd} & \mathbb{R}^m \arrow[dr, "g"] & \\ \mathbb{R}^n \arrow[ur, "f"] \arrow[rr, bend right, "g \circ f"'] & & \mathbb{R}^p \end{tikzcd}$$

$$(g \circ f)' = \underset{\substack{(p \times m)\\ \text{matrix}}}{g'(f(\vec{x}))} \; \underset{\substack{(m \times n)\\ \text{matrix}}}{f'(x)}$$

is $p \times n$ matrix

EXAMPLE



$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix} \qquad g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ 2u \\ v^2 \end{pmatrix}$$

Find $(g \circ f)'$ at $(2,3)$

(I) $f\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4+6+1 \\ 9+2 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \end{pmatrix}$; $g\begin{pmatrix} 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 22 \\ 22 \\ 121 \end{pmatrix}$

$(g \circ f)'(2,3) = g'\left(f\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = g'\begin{pmatrix} 11 \\ 11 \end{pmatrix}$ and $f'\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$g'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{array}{cc} u & v \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2v \end{pmatrix}} \end{array} \Rightarrow g'\begin{pmatrix} 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix}$$

$$f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{array}{cc} x & y \\ \multicolumn{2}{c}{\begin{pmatrix} 2x+y & x \\ 0 & 2y \end{pmatrix}} \end{array} \text{ so } f'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix}$$

MATH 223

Some Notes on Assignment 11

Exercises 17 and 18 in Chapter 4 and Problems A – C.

17. Show that the function f of one variable given by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } (x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable for all x but f' is not continuous at 0 so f is not continuously differentiable.

Solution: Wherever $x \neq 0$, $f(x)$ is the product and composition of differentiable functions, so it is differentiable. The interesting case is when $\frac{1}{x}$ is undefined at $x = 0$. To determine whether or not the function remains differentiable here we must inspect the limit of the difference quotient.

$$f'(0) = \lim_{x \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{x \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h}$$

if the limit exists. The function $\sin x$ is bounded above and below by 1, -1 for all x, so we can create the following upper and lower bounds:

$$-h \leq h \sin \frac{1}{h} \leq h \text{ so}$$

$$\lim_{h \to 0} -h \leq \lim_{h \to 0} h \sin \frac{1}{h} \leq \lim_{h \to 0} h$$

$$0 \leq \lim_{h \to 0} h \sin \frac{1}{h} \leq 0.$$

The derivative of f exists at at $x = 0$ and is equal to 0.

Using the rule for differentiation in the single-variable case we can find the derivative of f to be

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} : x \neq 0 \\ 0 : x = 0 \end{cases}$$

For $f'(x)$ to be continuous, the limit of $f'(x)$ as x approaches 0 must be zero. That is,

$$\lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

must equal 0; however, the first term in this limit will approach 0 as x gets small while the second term will achieve the values -1 and 1 on any neighbourhood of $x = 0$. The limit of the derivative of f does not then exist at $x = 0$.

18: Replace x^2 with x^3 in the previous exercise and determine if the resulting function is continuously differentiable everywhere.

Solution: To determine whether or not $f(x)$ is continuously differentiable we must first be certain that it is differentiable. In the case that $x \neq 0$, $f(x)$ is a product and composition