CLASS 12

HANDOUTS

NOTES ON ASSIGNMENT 11

ASSIGNMENT 12

$f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a}$ in $\mathbb{R}^n$ if there is an $m \times n$ matrix M such that

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x}) - f(a) - M(\vec{x}-\vec{a})}{|\vec{x}-\vec{t}|} = \vec{0}$$

$\boxed{\text{FOCUS ON } \mathbb{R}^m = 1}$

THM (4.2.1) If f is differentiable at $\vec{a}$, then $M = \nabla f$

THM (4.2.2) If all partials of f are continuous in an open set containing $\vec{a}$, then f is differentiable at $\vec{a}$

PARTIAL DERIVATIVE WITH RESPECT TO A VECTOR $\vec{V}$

$f(x,y) = 3x^2 + 5xy - 2y^2 + 1$

$\vec{a} = (2,-1)$

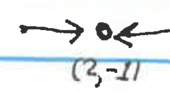
$f(\vec{a}) = f(2,-1) = +1$

$f_x(x,y) = 6x + 5y \qquad f_x(2,-1) = 7 \qquad \nabla f(2,-1) = (7,14)$

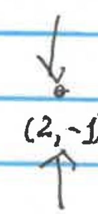
$f_y(x,y) = 5x - 4y \qquad f_y(2,-1) = 14$

$$f_x(\vec{a}) = \lim_{t\to 0} \frac{f(2+t,-1) - f(2,-1)}{t}$$

CAN WRITE AS $\displaystyle\lim_{t\to 0} \frac{f(\vec{a}+t(1,0)) - f(\vec{a})}{t}$



Similarly $\displaystyle f_y = \lim_{t\to 0} \frac{f(\vec{a}+t(0,1)) - f(\vec{a})}{t}$



What if we move toward $(2,-1)$ along vector $\vec{v} = (3,5)$

$$f_{\vec{v}}(\vec{a}) = \frac{\partial f(\vec{a})}{\partial \vec{v}} = \lim_{t\to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}$$

$$= \lim_{t\to 0} \frac{f(2+3t, -1+5t) - f(2,-1)}{t}$$

$$= \lim_{t\to 0} \frac{3(2+3t)^2 + 6(2+3t)(-1+5t) - 2(-1+5t)^2 + 1 - 1}{t}$$

$$f_{\vec{v}}(\vec{a}) = \lim_{t\to 0} \frac{52t^2 + 91t}{t} = \lim_{t\to 0} (52t + 91) = 91 \quad \square$$

NOTE

$$\nabla f(2,-1) \cdot \vec{v} = (7,14)\cdot(3,5) = 21 + 70 = 91$$

THEOREM: If $f: \mathbb{R}^n \to \mathbb{R}^1$ is differentiable at $\vec{a}$,

Then

$$f_{\vec{v}}(\vec{a}) = \nabla f(\vec{a}) \cdot v$$

where

$$f_{\vec{v}}(\vec{a}) = \lim_{t\to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}$$

Proof of Theorem.

(CASE 1): $\vec{v} = \vec{0}$ Then BOTH SIDES ARE 0

THUS, OK TO ASSUME $\vec{v} \neq \vec{0}$

By differentiability of f at $\vec{a}$, we have.

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - \nabla f(\vec{a})\cdot(\vec{x}-\vec{a})}{|\vec{x}-\vec{a}|} = 0$$

Set $\vec{x} = \vec{a} + t\,\vec{v}$

Then $\vec{x}\to\vec{a}$ is equivalent to $t\to 0$

and $\vec{x}-\vec{a} = t\,\vec{v}$

So

$$\lim_{t\to 0} \frac{f(\vec{a}+t\,\vec{v}) - f(a) - \nabla f(a)\cdot(t\vec{v})}{|t\,\vec{v}|} = 0$$

But $|t\,\vec{v}| = |t|\,|\vec{v}|$

can take $t>0$ (Why?) $\Rightarrow$ $|t\,\vec{v}| = t\,|\vec{v}|$

Now

$$\lim_{t\to 0}\left[\frac{f(\vec{a}+t\,\vec{v}) - f(a)}{t\,|\vec{v}|} - \frac{\cancel{t}\nabla f(a)\cdot\vec{v}}{\cancel{t}\,|\vec{v}|}\right] = 0$$

$$\Rightarrow \frac{1}{|\vec{v}|}\lim_{t\to 0}\left[\frac{f(\vec{a}+t\vec{v}) - f(a)}{t} - \frac{\nabla f(\vec{a})\cdot\vec{v}}{\cancel{|\vec{v}|}}\right] = 0$$

But $|\vec{v}|\neq 0$ so divide both sides by $1/|\vec{v}|$

$$\lim_{t\to 0}\left[\frac{f(\vec{a}+t\vec{v}) - f(a)}{t} - \nabla f(\vec{a})\cdot\vec{v}\right]$$

or

$$\underbrace{\lim_{t\to 0}\left[\frac{f(\vec{a}+t\vec{v}) - f(\vec{a})}{t}\right]} = \nabla f(\vec{a})\cdot\vec{v}$$

$$\Leftarrow \quad f_{\vec{v}}(\vec{a}) = \nabla f(\vec{a})\cdot\vec{v}$$

DIRECTIONAL DERIVATIVE

Suppose we want to measure the rate of change of f at $\vec{a}$ in the direction of $\vec{v}$.

We can't simply take $\nabla f(\vec{a}) \cdot \vec{v}$ because $\vec{v}, -2\vec{v}, 3\vec{v}$ all have the same direction but give different values for the dot product.

To get a STANDARD MEASURE, we approach along a vector of length 1, called a UNIT VECTOR

DEFINITION: If $|\vec{u}| = 1$, then $f_{\vec{u}}(\vec{a})$ is the DIRECTIONAL DERIVATIVE of f at $\vec{a}$ in the direction $\vec{u}$

Example $\vec{v} = (3,5)$ so $|\vec{v}| = \sqrt{3^2+5^2} = \sqrt{34}$

We let $\vec{u} = \left(\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right)$

Then directional derivative is $(7,14) \cdot \left(\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right) = \frac{91}{\sqrt{34}}$

$\Rightarrow$ RATE OF CHANGE IN DIRECTION $\vec{u}$

$$\nabla f(\vec{a}) \cdot \vec{u}$$
$$= |\nabla f(\vec{a})| \, |\vec{u}| \cos\theta$$
$$= |\nabla f(\vec{a})| \cos\theta \quad \text{since } |\vec{u}| = 1.$$

we maximize rate of change when $\cos\theta = 1$

$\Rightarrow \theta = 0 \Rightarrow$ PICK $\vec{u}$ IN direction of gradient