CLASS 9

HANDOUTS

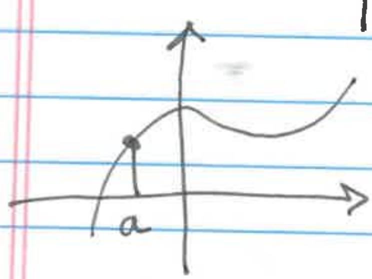
NOTES ON ASSIGNMENT 8

Assignment 9

ANNOUNCEMENTS

1. Office Hours Today: $12^{15}-1^{15}$
2. Abby's Drop-in Tutoring
   6–8 pm Sunday
3. EXAM 1
   MONDAY, 7 pm –
   NO TIME LIMIT
   NO BOOKS, NOTES, CALCULATOR, cell phones, etc
   Focus on chapters 2 and 3

Tangent Lines



2 forms

$$T(x) = f(a) + f'(a)(x-a)$$

$$T(s) = f(a) + s f'(a)$$

Example: $f(x) = x^3 - x + 3$ at $a = 2$

$f'(x) = 3x^2 - 1$.

Then $f(2) = 8 - 2 + 3 = 9$

$f'(2) = 3(4) - 1 = 11$

so $T(x) = 9 + 11(x-2)$

$T(s) = 9 + 11s$

Advantage of parameter form: CAN USE ON curves which ARE NOT graphs of functions

Tangent Planes to Surfaces

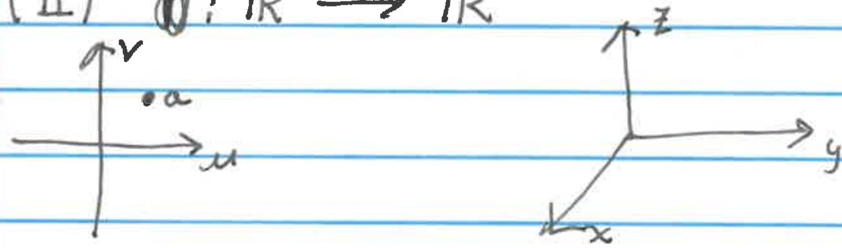
(I) $f: \mathbb{R}^2 \to \mathbb{R}^1$

$\vec{a}$ a point in $\mathbb{R}^2$

Tangent plane to graph of f at $(\vec{a}, f(\vec{a}))$

$$T(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

(II) $\sigma: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$



$$\sigma(u,v) = \left(f(u,v),\ g(u,v),\ h(u,v) \right)$$

$$\sigma_u(u,v) = (f_u,\ g_u,\ h_u)$$
$$\sigma_v(u,v) = (f_v,\ g_v,\ h_v)$$

Tangent Plane at $\sigma(\vec{a})$

$$\underset{1 \times 3}{\sigma(\vec{a})} + \underset{1 \times 2}{(u, v)} \underset{2 \times 3}{\begin{pmatrix} f_u & g_u & h_u \\ f_v & g_v & h_v \end{pmatrix}}$$

ordinary matrix multiplication

Better notation: write vectors vertically.

$$\sigma = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \qquad \sigma' = \begin{pmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{pmatrix}$$

Tangent plane.

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \underset{3 \times 1}{\sigma(\vec{a})} + \underset{(3 \times 2)}{\sigma'(a)} \underset{(2 \times 1)}{\begin{pmatrix} u \\ v \end{pmatrix}}$$

UTILITY

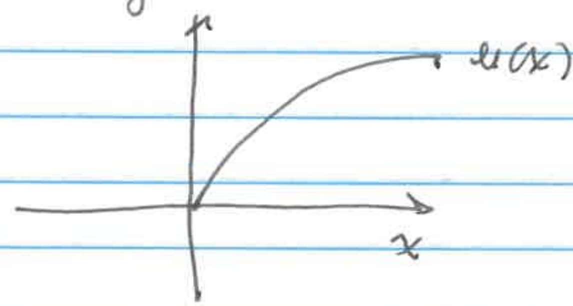
UTILITY = HAPPINESS, SATISFACTION, PLEASURE, USEFULNESS)

$$u(x), \quad x \geq 0$$

TYPICAL ASSUMPTION

u IS increasing, CONCAVE DOWN FUNCTION

"decreasing returns to scale"



Example $\quad u(x) = x^{1/3}$

$$u'(x) = \tfrac{1}{3}\, x^{-2/3} = \frac{1}{3\, x^{2/3}} > 0$$

$$u''(x) = -\tfrac{2}{9}\, x^{-5/3} \qquad < 0$$

EXAmple 2 goods: $u(x,y) = \sqrt[3]{xy}$

BUDGET CONSTRAINT $35x + 80y = D$

EACH UNIT of x COSTS \$35 } we have \$D to spend

EACH UNIT of y COSTS \$80

GOAL MAXIMIZE UTILITY

$$80y = D - 35x \Rightarrow y = \frac{D - 35x}{80}$$

$$\text{so } u(x,y) = f(x) = \sqrt[3]{\frac{x(D-35x)}{80}}$$

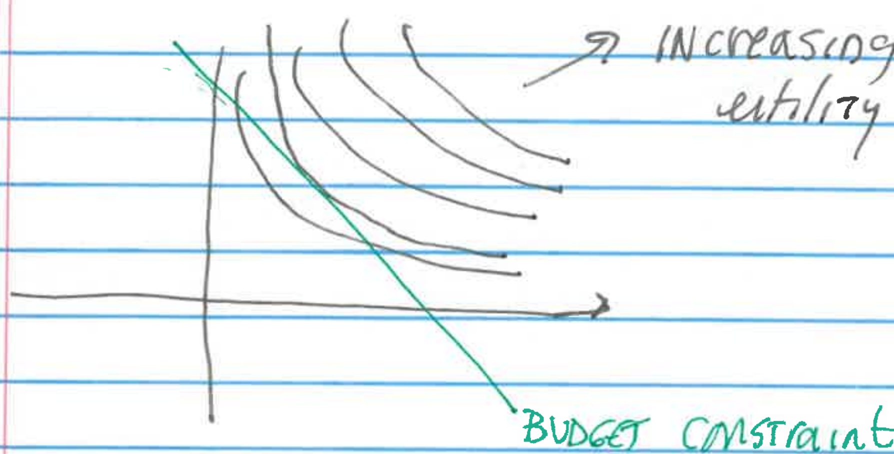
NOTE: f IS maximized when $\frac{x(D-35x)}{80}$ IS MAXIMIZED

$$G(x) = x(D - 35x) = Dx - 35x^2$$

$$G'(x) = D - 70x \qquad \Rightarrow \quad \text{MAX AT}$$

$$G''(x) = -70 \qquad\qquad x = D/70$$

$$\Rightarrow y = \frac{D - 35(D/70)}{80} = \frac{\frac{1}{2}D}{80} = D/160$$


INCREASING
UTILITY
BUDGET CONSTRAINT

Alexis Claude CLAIRAUT
5/7/1713 – 5/17/1765

CLAIRAUT'S THEOREM
EQUALITY of Mixed PARTIALS

If f_{xy} and f_{yx} are continuous at $\vec{a}$,

Then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$

$$f(x,y) = \begin{cases} 2xy\,\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

THEN IT TURNS out that

$$\left.\begin{aligned} f_{xy}(0,0) &= -2 \\ f_{yx}(0,0) &= +2 \end{aligned}\right\} \text{ INequality of Mixed Partials}$$

A Unified Treatment of Tangent Lines and Tangent Planes

Tangent Lines to Curves

Case I: $f:\mathbb{R}^1 \to \mathbb{R}^1$ (Coordinate representation of the curve)

Here the **graph** of f is the curve and $y = f(x)$ is equation of the curve

The tangent line to the graph of f at x = a is tangent line at $(a, f(a))$ and has equation $y = f(a) + f'(a)(x-a)$

Case 2: $f:\mathbb{R}^1 \to \mathbb{R}^2$ (Parametric Representation of the curve)

Here the ***image*** of f is the curve

We focus on the tangent line to the image of f.

Here $\mathbf{f}(t) = (f_1(t), f_2(t))$ and $\mathbf{f}'(t) = (f_1'(t), f_2'(t))$

Then: Tangent line is set of vectors of the form $\mathbf{f}(a) + \mathbf{f}'(a)t$

Case 3: $f:\mathbb{R}^1 \to \mathbb{R}^m$

Here the ***image*** of f is the curve

Now $\mathbf{f}(t) = (f_1(t), f_2(t), \ldots f_m(t))$ and $\mathbf{f}'(t) = (f_1'(t), f_2'(t), \ldots, f_m'(t))$

Again, Tangent line is set of vectors of the form $\mathbf{f}(a) + \mathbf{f}'(a)t$

Tangent Planes to Surfaces

Case 1: $f:\mathbb{R}^2 \to \mathbb{R}^1$ (coordinate representation of the curve)

Here the ***graph*** of f is the surface

Now $z = f(x,y)$ is the equation of the surface.

If (a,b) is a point in $\mathbb{R}^2$ in the domain of f, then the tangent plane to the surface at $(a,b,f(a,b))$ is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Note that we can write this equation in a number of equivalent ways:

$$z = f(a,b) + \left(f_x(a,b), f_y(a,b)\right) \bullet (x-a, y-b)$$

$$z = f(a,b) + \left(f_x(a,b), f_y(a,b)\right) \bullet \left((x,y) - (a,b)\right)$$

$$z = f(\mathbf{a}) + f'(\mathbf{a}) \bullet (\mathbf{x} - \mathbf{a}) \text{ where } \mathbf{a} = (a,b),\ \mathbf{x} = (x,y), \text{ and } f' = (f_x, f_y) = \nabla f$$

Case 2: $f:\mathbb{R}^2 \to \mathbb{R}^3$ (parametric representation of the surface)

Here the **image** of f is the surface

Now $f(u,v) = \left(f_1(u,v), f_2(u,v), f_3(u,v)\right)$

In vector form, with $\mathbf{a} = (u,v)$, and $f(\mathbf{a}) = \left(f_1(\mathbf{a}), f_2(\mathbf{a}), f_3(\mathbf{a})\right)$

Let $\dfrac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{pmatrix}$, $\dfrac{\partial f}{\partial v} = \begin{pmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial v} \end{pmatrix}$, and $f' = \nabla f = \left(\dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial v}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}$

The tangent plane to surface at $f(\mathbf{a})$ is the set of vectors of the form $f(\mathbf{a}) + f'(\mathbf{a})\begin{pmatrix} u \\ v \end{pmatrix}$

We can write out this vector formula as $\begin{pmatrix} f_1(\mathbf{a}) \\ f_2(\mathbf{a}) \\ f_3(\mathbf{a}) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial u}(\mathbf{a}) & \frac{\partial f_1}{\partial v}(\mathbf{a}) \\ \frac{\partial f_2}{\partial u}(\mathbf{a}) & \frac{\partial f_2}{\partial v}(\mathbf{a}) \\ \frac{\partial f_3}{\partial u}(\mathbf{a}) & \frac{\partial f_3}{\partial v}(\mathbf{a}) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{a}) + \frac{\partial f_1}{\partial u}(\mathbf{a})u + \frac{\partial f_1}{\partial v}(\mathbf{a})v \\ f_2(\mathbf{a}) + \frac{\partial f_2}{\partial u}(\mathbf{a})u + \frac{\partial f_2}{\partial v}(\mathbf{a})v \\ f_3(\mathbf{a}) + \frac{\partial f_3}{\partial u}(\mathbf{a})u + \frac{\partial f_3}{\partial v}(\mathbf{a})v \end{pmatrix}$

We may solve the first two equations for u and v in terms of x and y. Then substitute into third equation to get z in terms of x and y.

Summary: Equations for the Tangent Objects

	Curve	Surface
Coordinate Form	$y = f(a) + f'(a)(x-a)$	$z = f(\mathbf{a}) + f'(\mathbf{a}) \bullet (\mathbf{x}-\mathbf{a})$
Parametric Form	$\mathbf{f}(a) + \mathbf{f}'(a)t$	$f(\mathbf{a}) + f'(\mathbf{a})\begin{pmatrix} u \\ v \end{pmatrix}$
	Tangent is a Line	**Tangent is a Plane**