CLASS 2

HANDOUT

Assignment 2

Announcement

Office Hours Today: 12:15 - 1:30

TUTOR: Abby Truex, 6-8 PM MW, 311 MUNROE

Analog of Straight Line in Higher Dimensions

LINE: $ax + by = c$

$a_1 x_1 + a_2 x_2 = c$

PLANE $a_1 x_1 + a_2 x_2 + a_3 x_3 = c$

$ax + by + cz = d$

HYPERPLANE: $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = d.$

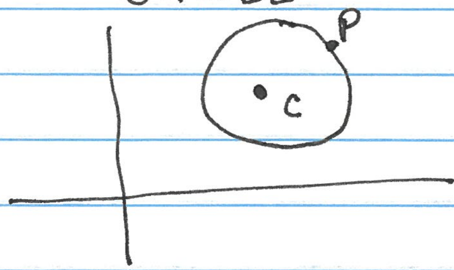
OTHER important CURVES: CIRCLES AND ELLIPSES and their counterparts in Higher Dimensions

Recall: graph of $f: \mathbb{R}^1 \to \mathbb{R}^1$ is a curve in the plane.

BUT: NOT EVERY CURVE IN THE PLANE IS THE GRAPH of such a function.

(VERTICAL LINE TEST)

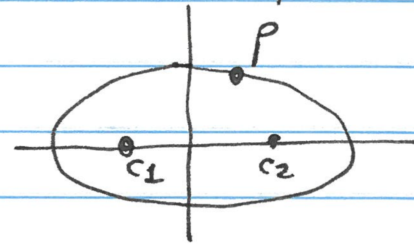
CIRCLE



Set of all points a fixed distance from a fixed point (center)

distance $(P, C) = r$

ELLIPSE



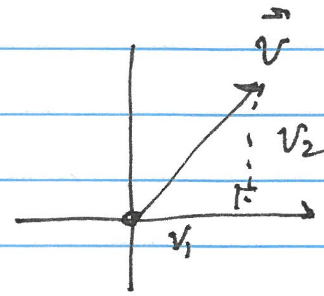
Set of all points, sum of distances to pair of fixed points is constant

$d(P, C_1) + d(P, C_2) = r$

DISTANCE IN $\mathbb{R}^n$

MAGNITUDE of a vector $\vec{v}$

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$$



Where does this come from?

Consider $\vec{v} = (\vec{v}_1, \vec{v}_2)$ in $\mathbb{R}^2$

Then $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2$

and $\sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2}$ PYTHAGOREAN THEOREM

DISTANCE BETWEEN $\vec{x}$ and $\vec{y}$ = $|\vec{x} - \vec{y}|$

EXAMINE CIRCLE IN PLANE

CENTER $\vec{C} = (a, b)$ radius r

point $\vec{P} = (x, y)$

Defining Relationship: $d(P, C) = r$

THIS Means

$$|\vec{P} - \vec{C}| = r$$

$$|(x-a, y-b)| = r$$

$$\sqrt{(x-a)^2 + (y-b)^2} = r$$

$$(x-a)^2 + (y-b)^2 = r^2$$

MULTIPLY out

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

$$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$$

FORM

$$x^2 + y^2 + Ax + By = C$$

CAN we go backwards:

$$x^2 + y^2 - 6x + 16y = 71$$

complete squares in x and in y

$$x^2 - 6x \quad + \quad y^2 + 16y \quad = 71$$

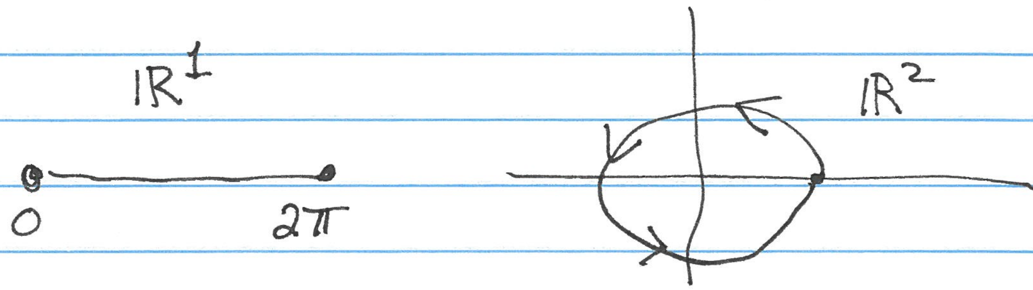
$$x^2 - 6x + 9 \quad + \quad y^2 + 16y + 64 \quad = 71 + 9 + 64$$

$$(x-3)^2 + (y+8)^2 = 144 = 12^2$$

CIRCLE AS IMAGE OF A FUNCTION $\vec{f}: \mathbb{R}^1 \to \mathbb{R}^2$

Example $\vec{f}(t) = (\cos(t), \sin(t))$, $0 \le t \le 2\pi$

PARAMETRIZATION WITH PARAMETER t



Example
$$\begin{cases} x = 12 \cos t + 3 \\ y = 12 \sin t - 8 \end{cases}$$

so

$$\begin{cases} x - 3 = 12 \cos t \\ y + 8 = 12 \sin t \end{cases}$$

$$(x-3)^2 + (y+8)^2 = 12^2$$

$$\vec{f}(t) = (12 \cos t + 3, 12 \sin t - 8)$$

ELLIPSE

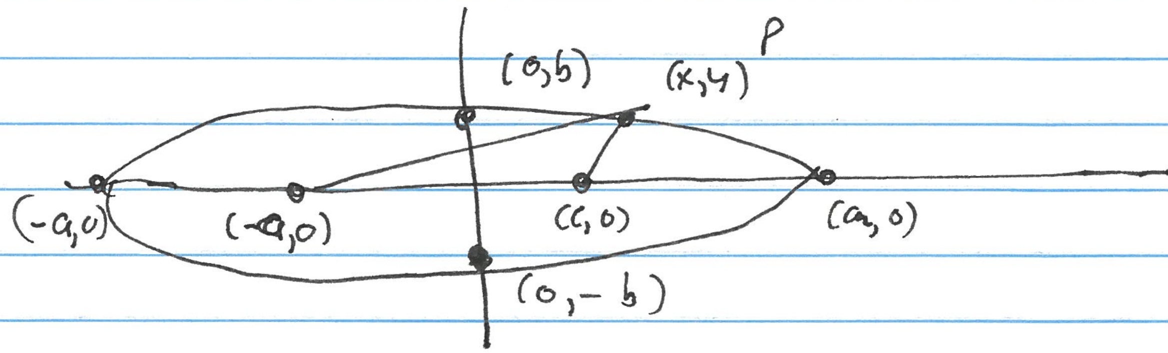
STANDARD ELLIPSE

HORIZONTAL AND VERTICAL AXES

CENTER AT $(0,0)$

FOCI $(\pm c, 0)$

VERTICES $(\pm a, 0)$ and $(0, \pm b)$



$(a,0)$ DISTANCE from $(c,0)$ + distance from $(-c,0)$

$$= (a-c) + (a-(-c)) = a-c+a+c$$

$$= 2a$$

$(0,b)$ DISTANCE from $(c,0)$ + DISTANCE FROM $(-c,0)$

$$\sqrt{c^2+b^2} + \sqrt{c^2+b^2} = 2\sqrt{c^2+b^2}$$

THUS $$2\sqrt{c^2+b^2} = 2a^2$$

$$\Rightarrow \sqrt{c^2+b^2} = a^2$$

$$\Rightarrow c^2+b^2 = a^2 \Rightarrow c^2 = a^2 - b^2$$

(x,y) DISTANCE from $(c,0)$ + DISTANCE $(-c,0) = 2a$

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

MUCH ALGEBRA YIELDS

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametrization $\vec{g}(t) = (a\cos t,\ b\sin t),\ 0 \le t \le 2\pi$

THE ALGEBRA:

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

WRITE AS

$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

SQUARE BOTH SIDES

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

EXPAND

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2$$

$$\Rightarrow \quad 2cx = 4a^2 - 4a\sqrt{(x-c)^2+y^2} - 2cx$$

OR $4cx - 4a^2 = -4a\sqrt{(x-c)^2+y^2}$

DIVIDE by 4:

$$cx - a^2 = -a\sqrt{(x-c)^2+y^2}$$

Square again

$$c^2x^2 - 2a^2cx + a^4 = a^2\left[x^2 - 2cx + c^2 + y^2\right]$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

WRITE AS

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

DO A LITTLE FACTORING

$$x^2(a^2-c^2) + a^2y^2 = a^2(a^2-c^2)$$

BUT $a^2 - c^2 = b^2$ so

$$x^2b^2 + a^2y^2 = a^2b^2$$

DIVIDE by a^2b^2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$