CLASS 14

HANDOUTS

NOTES ON ASSIGNMENT 13

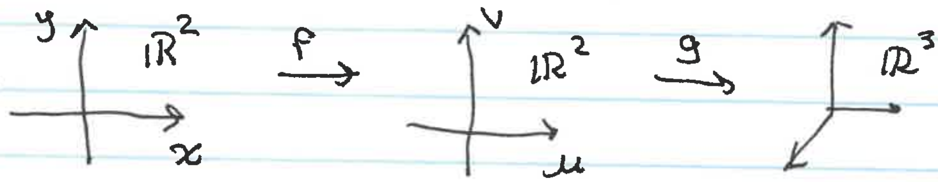
ASSIGNMENT 14

CHAIN RULE

$$\begin{tikzcd} & \mathbb{R}^m \arrow[dr, "g"] & \\ \mathbb{R}^n \arrow[ur, "f"] & & \mathbb{R}^p \end{tikzcd}$$

$$(g \circ f)' = \underset{\substack{(p \times m) \\ \text{MATRIX}}}{g'(f(\vec{x}))}\ \underset{\substack{(m \times n) \\ \text{MATRIX}}}{f'(\vec{x})}$$

<u>EXAMPLE</u>



$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix} \qquad g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ 2u \\ v^2 \end{pmatrix}$$

FIND $(g \circ f)'$ at $(2,3)$

(I) $f(2,3) = \begin{pmatrix} 2^2 + (2)(3) + 1 \\ 3^2 + 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \end{pmatrix} \Rightarrow$ THUS we want $g'\begin{pmatrix} 11 \\ 11 \end{pmatrix}$

NOW $g'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{array}{c} \begin{array}{cc} u & v \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2v \end{pmatrix} \end{array}$ so $g'\begin{pmatrix} 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix}$

BUT $f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & x \\ 0 & 2y \end{pmatrix}$ so $f'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix}$

THUS $(g\circ f)'\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1 & 1\\ 2 & 0\\ 0 & 22\end{pmatrix}\begin{pmatrix}7 & 2\\ 0 & 6\end{pmatrix} = \begin{pmatrix}7 & 8\\ 14 & 4\\ 0 & 132\end{pmatrix}$

Here we can actually check by direct COMPUTATION

$$g\big(f(x,y)\big) = g\begin{pmatrix}x^2+xy+1\\ y^2+2\end{pmatrix} = \begin{pmatrix}x^2+xy+1+y^2+2\\ 2x^2+2xy+2\\ y^4+4y^2+4\end{pmatrix}$$

THEN

$$(g\circ f)'\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+y & x+2y\\ 4x+2y & 2x\\ 0 & 4y^3+8y\end{pmatrix}$$

SO

$$(g\circ f)'\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}7 & 8\\ 14 & 4\\ 0 & 132\end{pmatrix}$$

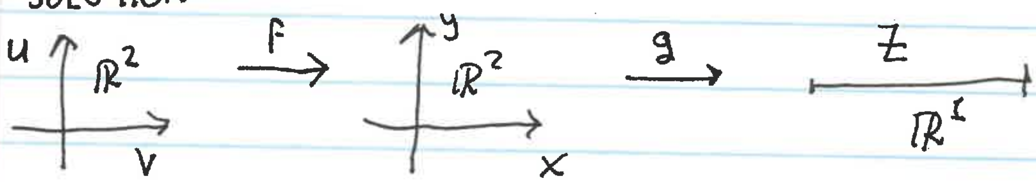
EXAMPLE: Suppose $x = u^2 - v^2$

$y = 2uv$

and $z = g(x, y)$ for some real-valued function g which is differentiable.

SHOW $(z_u)^2 + (z_v)^2 = 4(u^2+v^2)\left[(z_x)^2 + (z_y)^2\right]$

SOLUTION



where $f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^2 - v^2 \\ 2uv \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

THEN $f'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$ and $g'\begin{pmatrix} x \\ y \end{pmatrix} = (g_x, g_y) = (z_x, z_y)$

Now $(g \circ f)' = g'(f)\, f' = (z_x, z_y)\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$

$$\left\{ = (z_x, z_y)\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right\}$$

$$\Big(\underbrace{2u z_x + 2v z_y}_{z_u},\ \underbrace{-2v z_x + 2u z_y}_{z_v}\Big)$$

$$\begin{aligned}
(z_u)^2 + (z_v)^2 &= 4u^2(z_x)^2 + 8uv\, z_x z_y + 4v^2(z_y)^2 \\
&\quad + 4v^2(z_x)^2 - 8uv\, z_x z_y + 4u^2(z_y)^2 \\
&= 4u^2(z_x^2 + z_y^2) + 4v^2(z_x^2 + z_y^2) \\
&= 4(u^2+v^2)\left([z_x]^2 + [z_y]^2\right)
\end{aligned}$$

IMPLICIT DIFFERENTIATION

Example FIND slope of Tangent Line to the graph of

$$4x^2 + 5y^2 = 61 \quad \text{at } (2,3)$$

CHECK THAT POINT LIES ON CURVE

$$4(2^2) + 5(3^2) = 16 + 45 = 61.$$

(A) DIRECT SOLUTION

$$5y^2 = 61 - 4x^2$$

$$y^2 = \frac{61-4x^2}{5}$$

$$y = \sqrt{\frac{61-4x^2}{5}} = \left(\frac{61-4x^2}{5}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{61-4x^2}{5}\right)^{-1/2}\left(-\frac{8x}{5}\right)$$

EVALUATE WHEN $x = 2$

$$\frac{1}{2}\left(\frac{45}{5}\right)^{-1/2}\left(-\frac{16}{5}\right) = -8/15$$

(B) CLASSIC IMPLICIT DIFFERENTIATION

TREAT y as an unknown FUNCTION of x and differentiate: $8x + 10y\, dy/dx = 0$

$$10y\, dy/dx = -8x$$

$$dy/dx = -\frac{8x}{10y} = -\frac{4x}{5y} = \frac{-4(2)}{5(3)} = \frac{-8}{15}$$

(C) CONSIDER $f(x,y) = 4x^2 + 5y^2$

THEN $(2,3)$ LIES ON LEVEL CURVE $f(x,y) = 61$

THEN $\nabla f(2,3)$ IS NORMAL TO CURVE

Now $\nabla f(x,y) = (8x, 10y)$ so $\nabla f(2,3) = (16, 30)$

and slope of $\nabla f(2,3)$ is $30/16 = 15/8$

$\Rightarrow$ slope of tangent line is $-8/15$