CLASS 10

HANDOUTS

NOTES ON ASSIGNMENT 9

Assignment 10

Differentiation

EXAM: TONIGHT

MATH 223 A: AXINN 229

MATH 223 B: AXINN 232

TODAY: BEGIN CHAPTER 4

TOPIC: DIFFERENTIABILITY

START WITH: $f: \mathbb{R}^n \to \mathbb{R}^1$ REAL-VALUED

EVENTUALLY $\vec{F}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. VECTOR-VALUED.

DERIVATIVE AT A POINT TURNS OUT TO BE

$m \times n$ matrix

BUT FIRST... LIMITS AND CONTINUITY

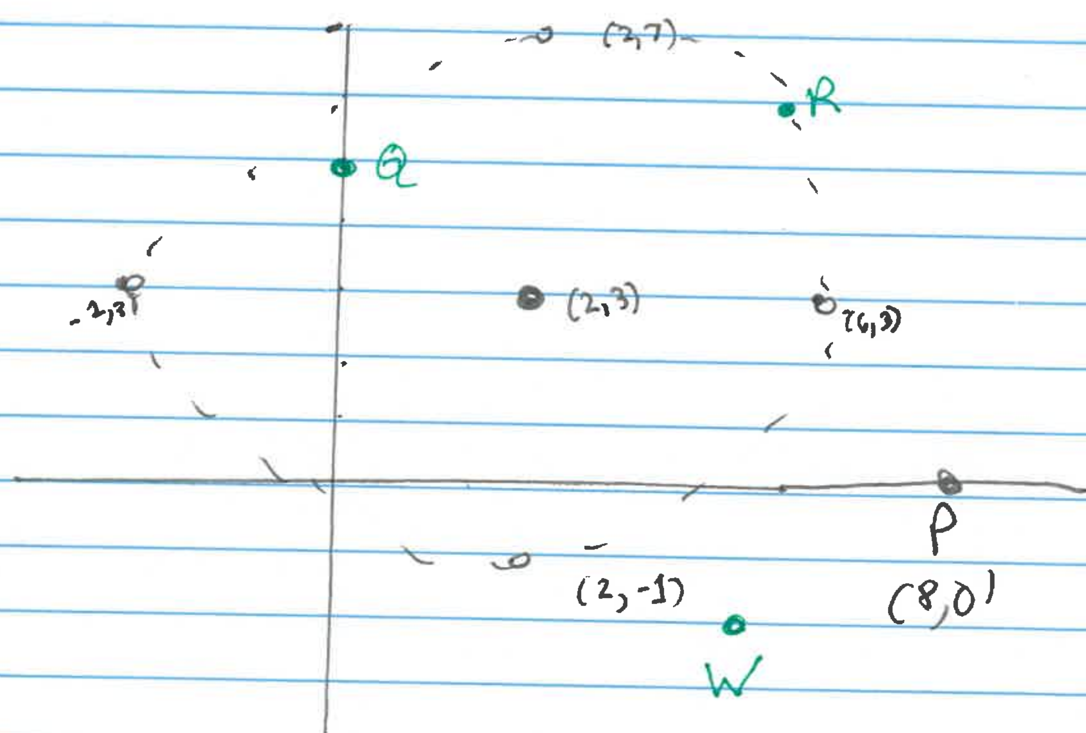
# LIMITS AND CONTINUITY

## Preliminary Concepts

| | |
|---|---|
| Open Sets | Interior Point |
| Closed Set | Boundary Point |
| Limit Point | Neighborhood |

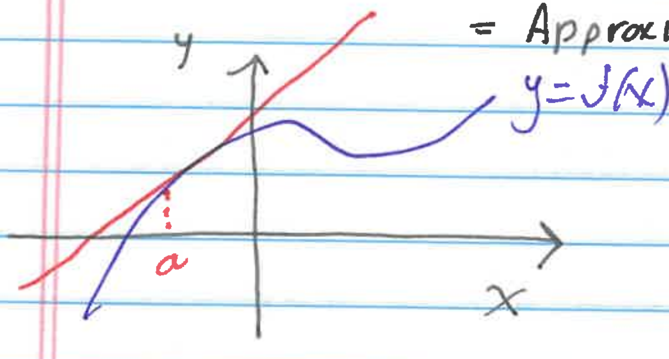
EXAMPLE: Let $S = \{ |\vec{x} - (2,3)| < 4 \} \cup \{ (8,0) \}$

| POINT | INTERIOR POINT | LIMIT POINT | BOUNDARY |
|---|---|---|---|
| Q | YES | YES | NO |
| R | NO | YES | YES |
| P | NO | NO | YES |
| W | NO | NO | NO |



DIFFERENTIABILITY = LOCAL LINEARITY

= Approximarable by Tangent Object




Let $m = f'(a)$

$$f(x) \approx f(a) + f'(a)\,(x-a)$$

OR $f(x) - f(a) \approx m\,(x-a)$

OR $f(x) - f(a) - m\,(x-a) \approx 0$

$$\lim_{x \to a} \frac{f(x) - f(a) - m\,(x-a)}{|x-a|} = 0$$

Generalizing : $f: \mathbb{R}^n \to \mathbb{R}^m$

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - M\,(\vec{x} - \vec{a})}{|\vec{x} - \vec{a}|} = \vec{0}$$

for some $m \times n$ matrix $M$

Special case: $m = 1$, $n = 2$

$M$ is $1 \times 2$ matrix

$$\nabla f = (f_x, f_y)$$

Example $f(x,y) = x^2 + 2xy - y^2$ at $(-1, 2)$

$f(-1,2) = -7$

$f_x(x,y) = 2x + 2y \Rightarrow \nabla f(-1,2) = (2, -6)$

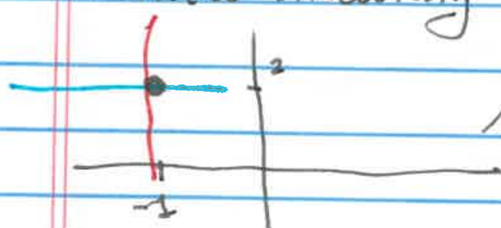
$f_y(x,y) = 2x - 2y$

Equation of tangent plane

$$z = -7 + (2,-6)\cdot(x+1, y-2) = -7 + 2x + 2 - 6y + 12$$

$$z = 7 + 2x - 6y$$

Review meaning of $f_x(-1,2) = 2$ and $f_y(-1,2) = -6$




what is range of change of $f$ at $(-1,2)$ if we approach along direction given by $\vec{v} = (3,4)$?

$$f_{\vec{v}} = \lim_{t\to 0} \frac{f(-1+3t, 2+4t) - f(-1,2)}{t}$$

$$= \lim_{t\to 0} \frac{(-1+3t)^2 + 2(-1+3t)(2+4t) - (2+4t)^2 - (-7)}{t}$$

$$= \lim_{t\to 0} \frac{17t^2 - 18t}{t} = \lim_{t\to 0}(17t - 18) = -18$$

NOTE: $(\nabla f)\cdot\vec{v} = (2,-6)\cdot(3,4)$

$$= (2)(3) + (-6)(4)$$

$$= 6 - 24 = -18$$

IS THAT A COINCIDENCE?

Multivariable Calculus Chapter :

DIFFERENTIABLITY

*Section 1: Limits and Continuity*

Part A: Neighborhoods

*Definition*: A **δ-ball** in $\mathbb{R}^n$ with radius $\delta > 0$ and center $x_o$ is the set of all points $\mathbf{x}$ in $\mathbb{R}^n$ such that $|\mathbf{x} - x_o| < \delta$.

*Definition:* A **neighborhood** of a point $\mathbf{p}$ in $\mathbb{R}^n$ is a δ-ball with center at $\mathbf{p}$ for some $\delta > 0$.

*Definition:* If $S$ is a set in $\mathbb{R}^n$ and $\mathbf{p}$ is a member of $S$, then $\mathbf{p}$ is an **interior point** of $S$ if there is some neighborhood $N$ of $\mathbf{p}$ entirely contained in $S$.

*Definition:* A set $S$ in $\mathbb{R}^n$ is **open** if every point of $S$ is an interior point of $S$.

*Definition:* If $S$ is a set in $\mathbb{R}^n$, then $\mathbf{p}$ is a **limit point** of $S$ if every neighborhood of $\mathbf{p}$ contains a point $\mathbf{q}$ of $S$ which is distinct from $\mathbf{p}$.

**Note:** (1) $\mathbf{p}$ is a limit point of $S$ if for every given $\delta > 0$, there is a point $\mathbf{q}$ in $S$ such that $0 < |\mathbf{p} - \mathbf{q}| < \delta$.

(2) $\mathbf{p}$ can be a limit point of $S$ without belonging to $S$.

*Definition:* A **boundary point** of a set $S$ in $\mathbb{R}^n$ is a point $\mathbf{p}$ such that every neighborhood of $\mathbf{p}$ contains both a point in $S$ and a point not in $S$. The **boundary** of a set is the set of all boundary points of $S$.

Definition: A set in $\mathbb{R}^n$ is **closed** if it contains all of its boundary points.

## Part B: Limits

*Definition:* Suppose $f$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $\mathbf{y}_o$ be a point in $\mathbb{R}^m$ and $\mathbf{x}_o$ a limit point of the domain of $f$. Then $\mathbf{y}_o$ is the **limit** of $f$ at $\mathbf{x}_o$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{y}_o| < \varepsilon$ whenever $\mathbf{x}$ is in the domain of $f$ and satisfies $0 < |\mathbf{x} - \mathbf{x}_o| < \delta$.

**Notation:** We write $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{y}_0$

**Note:** An alternative way to phrase the last sentence in the definition of limit is: $\mathbf{y}_o$ is the limit of $f$ at $\mathbf{x}_o$ if for every neighborhood $N_y$ of $\mathbf{y}_o$, there is a neighborhood $N_x$ of $\mathbf{x}_o$ such that $f(\mathbf{x})$ lies in $N_y$ whenever $\mathbf{x}$ is a member of $N_x$ other than $\mathbf{x}_o$.

*Theorem*: Limits are unique: If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{y}_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{y}_2$ then $\mathbf{y}_1 = \mathbf{y}_2$.

*Theorem*: Given $f: \mathbb{R}^n \to \mathbb{R}^m$ with coordinate functions $f_1, \ldots, f_m$ and a point $\mathbf{y}_o = (y_1, \ldots, y_m)$ in $\mathbb{R}^m$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$ if and only if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = y_i, \text{ for } i = 1, \ldots, m$$

## Part C: Continuity

*Definition:* A function $f$ is continuous at $x_o$ if

(a) $x_o$ is in the domain of $f$

(b) $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$

Other ways of phrasing continuity are

(a) $f$ is continuous at $x_o$ if for each neighborhood N of $f(x_o)$, there is a neighborhood $M$ of $x_o$ such that f takes each of point of M into some point of $N$.

(b) $f$ is continuous at $x_o$ if for every open set $U$ containing $f(x_o)$, the set of points $f$ maps into $U$ is open.

*Theorem*: A vector function is continuous at a point if and only if all its coordinate functions are continuous there.

*Theorem*: The functions $P_k := \mathbb{R}^n \to \mathbb{R}$ where $P_k(x_1,...,x_n) = x_k$ are continuous for $k$ = 1,2...,n. The function $P_k$ is called the *k*th coordinate projection.

*Theorem*: The functions $S : \mathbb{R}^2 \to \mathbb{R}$ and $M$: $\mathbb{R}^2 \to \mathbb{R}$ defined by $S(x,y) = x + y$ and $M(x,y) = xy$ are continuous.

*Theorem*: If $f$: $\mathbb{R}^n \to \mathbb{R}^m$ and $g$: $\mathbb{R}^m \to \mathbb{R}^p$ are continuous then the composition $g \circ f$ is continuous wherever it is defined.

*Definition:* A fiction $f$: $\mathbb{R}^n \to \mathbb{R}^m$ is a **linear** function if each coordinate function has the form $f_k(x_1,..x_n) = a_{k1}x_1 + ... + a_{kn}x_n$ for some scalars $a_{kj}$.

*Theorem*: A linear function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous.