CLASS 15

HANDOUTS

NOTES ON ASSIGNMENT 14

ASSIGNMENT 15

TODAY

BIT MORE ON IMPLICIT DIFFERENTIATION

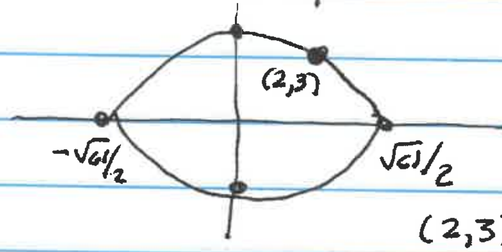
CHANGE OF VARIABLES

GRADIENT FIELDS

IMPLICIT DIFFERENTIATION (WILL REVISIT LATER)

FIND SLOPE OF TANGENT LINE TO THE GRAPH OF

$$4x^2 + 5y^2 = 61 \text{ at } (2,3)$$



Level curve of $F(x,y) = 4x^2 + 5y^2$

piece of curve around $(2,3)$ is graph of $y = f(x)$ for some unknown f.

We want $f'(2)$

$$\mathbb{R}^1 \xrightarrow{G} \mathbb{R}^2 \xrightarrow{F} \mathbb{R}^1$$

Define $G(x): \mathbb{R}^1 \to \mathbb{R}^2$ by $G(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}$ so $G'(x) = \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$ is tangent vector.

Then

$$(F \circ G)(x) = 61 \quad \text{for all } x$$

TAKE DERIVATIVE using chain Rule.

$$F'(G(x))\, G'(x) \equiv 0$$

$$\nabla F(G(x)) \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} = 0 \quad \text{gradient} \perp \text{Tangent}$$

$$(8x, 10y)\begin{pmatrix} 1 \\ f'(x) \end{pmatrix} = 0 \quad \text{set } x = 2,\ y = 3$$

$$(16, 30)\begin{pmatrix} 1 \\ f'(2) \end{pmatrix} = 0. \text{ solve for } f'(2)$$

CHANGE OF VARIABLE

Example $\int (10x + 15)^{1/3}\, dx$

MAKE change of variable $u = 10x + 15$

Solve for x: $x = \frac{u-15}{10}$] Key STEP *

Then $dx = \frac{1}{10}\, du$

Integral becomes

$$\int (10x+15)^{1/3}\, dx = \int u^{1/3} \tfrac{1}{10}\, du = \tfrac{1}{10}\int u^{1/3}\, du$$

$$= \tfrac{1}{10}\,\tfrac{3}{4}\, u^{4/3} + C$$

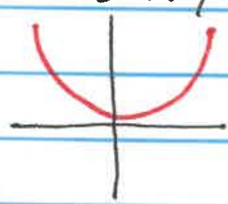
$$= \tfrac{3}{40}\,[10x+15]^{4/3} + C$$

(*) MUST BE ABLE TO INVERT THE SUBSTITUTION

CHANGE OF VARIABLE SHOULD BE INVERTIBLE, A ONE-TO-ONE FUNCTION.

NOTE: NOT EVERY FUNCTION IS INVERTIBLE

$y = x^2$



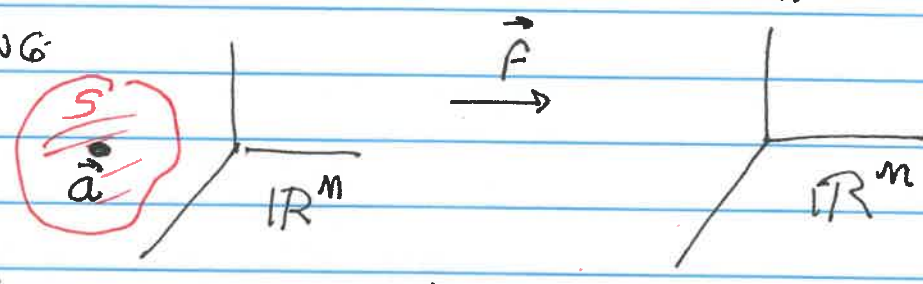
CAN'T solve UNAMBIGUOUSLY for x IN TERMS of y. globally

$x = \pm\sqrt{y}$. BUT CAN solve LOCALLY

HORIZONTAL LINE TEST

INVERSE FUNCTION THEOREM

SETTING:



IF: $\vec{a}$ is point in $\mathbb{R}^n$

S is open set containing $\vec{a}$

f is continuously differentiable on S

AND DERIVATIVE MATRIX $f'(\vec{a})$ IS INVERTIBLE

THEN

There is a Neighborhood N of $\vec{a}$ on which:

f^{-1} is defined ~~ad~~ and

$$\left(f^{-1}(x)\right)' = \left[f'(x)\right]^{-1} \quad \text{for all } \vec{x} \text{ in } N$$

EXAMPLE $f(x, y) = (\cos x,\ x\cos x - y)$

THEN

$$J = f'(x,y) = \begin{pmatrix} -\sin x & 0 \\ \cos x - x\sin x & -1 \end{pmatrix}$$

$\det J = \sin x$ so we have invertibility if $x \neq 0, \pi$

It is invertible at $x = \pi/6$

$$(f^{-1})' = J^{-1} = \begin{pmatrix} \frac{-1}{\sin x} & 0 \\ \frac{x\sin x - \cos x}{\sin x} & -1 \end{pmatrix}$$

At $x = \pi/6,\ y = 2$

$$f(\pi/6, 2) = \left(\sqrt{3}/2,\ \frac{\pi}{6}\frac{\sqrt{3}}{2} - 2\right)$$

and

$$(f^{-1})' = \begin{pmatrix} -1/\tfrac{1}{2} & 0 \\ \frac{\frac{\pi}{6}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}}{1/2} & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ \frac{\pi}{6} - \sqrt{3} & -1 \end{pmatrix}$$

GRADIENT FIELD

If $f: \mathbb{R}^n \to \mathbb{R}^1$, then $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$

Example 1 $f(x,y) = (x^2 \sin y)$

Then $\nabla f(x,y) = (2x \sin y, x^2 \cos y) = (f_x, f_y)$

NOTE

$$\left.\begin{aligned} f_{xy} &= 2x \cos y \\ f_{yx} &= 2x \cos y \end{aligned}\right\} \text{ EQUALITY of Mixed Partials.}$$

Example 2. $F(x,y) = (y, 2x)$ ~~is~~ IS NOT A GRADIENT FIELD

Why: if $F = \nabla f$, Then

$f_x(x,y) = y \Rightarrow f_{xy}(x,y) = 1$ NOT
$f_y(x,y) = 2x \Rightarrow f_{yx}(x,y) = 2$ EQUAL

What if we TRY TO build f by "PARTIAL INTEGRATION"?

$f_x = y$

$\Rightarrow f = xy + G(y)$ where G is a function only of y.

$\Rightarrow f_y = x + G'(y)$

NEED $G'(y) = x$ $\boxed{\text{IMPOSSIBLE}}$

WORKBACKWARDS ON Example 1

$f_x = 2x \sin y$

$\Rightarrow f = x^2 \sin y + G(y)$

Then $f_y = x^2 \cos y + G'(y)$

NEED $G'(y) = 0$ so Take any constant C.

so

$f(x,y) = x^2 \sin y + C$