MATH 223

Hints and Answers for Assignment 19

Exercises 18ad, 19ac, 25 and 26 in Chapter 5

18: (a) The third order partial derivatives of f are all 0, and the Hessian matrix at $\left(\frac{-14}{3}, \frac{-16}{3}\right)$ is positive definite; thus, this point is a relative minimum. Because f is continuous and there are no other critical points, f achieves its minimum value at this point. The minimum value of f is $f\left(\frac{-14}{3}, \frac{-16}{3}\right) = 0$.
(d) The Hessian matrix has one positive eigenvalue and one negative eigenvalue; it is neither positive definite nor negative definite. The critical point $(-2, 3)$ is a saddle point and not an extreme of f. As there are no other critical points of f and the function is continuous for all (x, y), there must be no highest and lowest values of $f(x, y) = z$.

19: (a) A critical point of f is any point (x, y) such that $\nabla f(x, y) = (0, 0)$. The gradient of f is $\nabla f(x, y) = (3x^2, -3y^2)$. The origin is the only critical point. Evaluated at the origin the Hessian of f is

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The function f then has a saddle point at the origin.

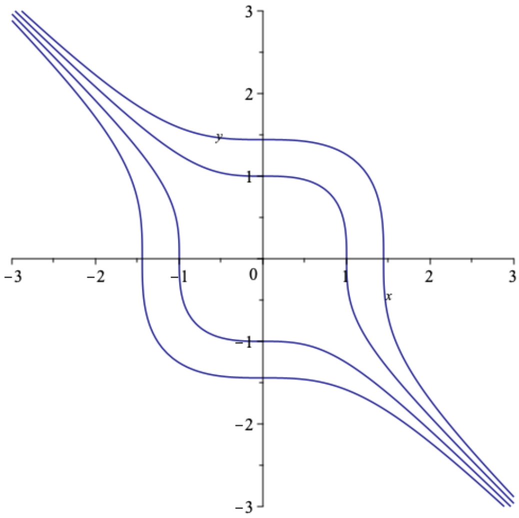

Figure 1: Level curves for the function $f(x, y) = x^3 - y^3$.

(c) If $f(x, y) = \frac{1}{e^{x^2+y^2}}$ then the gradient of f is $\nabla f = \left(\frac{-2x}{e^{e+y^2}}, \frac{-2y}{e^{x^2+y^2}}\right)$. We can see that the partial derivative f_x will only be 0 when $x = 0$ and the partial derivative f_y will only be 0 when $y = 0$; thus, the only critical point of f is the origin.
The Hessian matrix of f is

$$H = \begin{pmatrix} \frac{4x^2-2}{e^{x^2+y^2}} & \frac{4xy}{e^{2(x^2+y^2}} \\ \frac{4y^2-2}{e^{x^2+y^2}} & \frac{4xy}{e^{2(x^2+y^2}} \end{pmatrix}.$$

The origin is a local maximum of f.

25: The functions $F(x, y, \lambda)$ has a critical point when $x = \frac{c\alpha}{a}$ and $y = \frac{c\beta}{b}$; thus f is maximized at this point.

26: We have $y = \frac{d\beta}{b}$, and $z = \frac{d\gamma}{c}$. The function F has a critical point at $\left(\frac{d\alpha}{a}, \frac{d\beta}{b}, \frac{d\gamma}{c}, \lambda\right)$; therefore, f reaches its maximum with respect to the constraint function at $\left(\frac{d\alpha}{a}, \frac{d\beta}{b}, \frac{d\gamma}{c}\right)$.