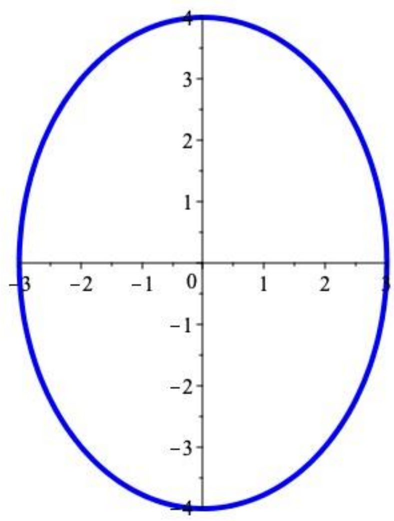
MATH 223 Multivariable Calculus
Final Examination Solutions
December 14 and 15, 2021

Throughout this exam E denotes the ellipse in the (xy)-plane defined by the equation $\frac{x^2}{9}+\frac{y^2}{16}=1$ and R is the set of points on or inside the ellipse.

1. 1Let E be the ellipse in the plane defined by the equation $\frac{x^2}{9}+\frac{y^2}{16}=1$ and let R be the set of points contained inside or on the ellipse.

(a) Sketch a picture of E.



(b) Show that E can be parametrized by $\boldsymbol{g}(t)=(3\cos t, 4\sin t), 0\le t\le 2\pi$.
with $x = 3\cos t$ and $y = 4 \sin t$, we have $x^2 = 9\cos^2$ t, $y^2 = \sin^2 t$ so that

$$\frac{x^2}{9}+\frac{y^2}{16}= cos^2 t + sin^2 = 1$$

(c) Let $P\left(\frac{9}{5},\frac{16}{5}\right)$ be the point. Find an equation for the tangent line to the ellipse at this point P.

Method I: Use classic implicit differentiation: Differentiate $\frac{x^2}{9}+\frac{y^2}{16}=1$ with respect to x:

$\frac{2x}{9}+\frac{2y}{16}\frac{dy}{dx}=0$. Solve solve $\frac{dy}{dx}=-\frac{\frac{2x}{9}}{\frac{2y}{16}}=\frac{-2x}{9}\frac{16}{2y}=-\frac{16}{9}\frac{x}{y}$.

At P, $\frac{x}{y}=\frac{\frac{9}{5}}{\frac{16}{5}}=\frac{9}{16}$, so slope of tangent line is $-\frac{16}{9}\frac{9}{16}=-1$.

Equation of Tangent Line: $y-\frac{16}{5}=-\left(x-\frac{9}{5}\right)$ which becomes $y=5-x$.

Method II: At P, $\frac{9}{5}=3\cos t, \frac{16}{5}=4\sin t$, so $\sin t=\frac{4}{5}, \cos t=\frac{3}{5}$

Use $g'(t)=(-3\sin t, 4\cos t)$. At P, $=g'(t)=\left(-\frac{12}{5},\frac{12}{5}\right)$.

We can also write the tangent line in the form $\left(\frac{9}{5},\frac{16}{5}\right)+t\left(-\frac{12}{5},\frac{12}{5}\right)$.

(d) Find a unit vector normal to the ellipse at this point P.
Tangent vector has slope -1 so Normal vector has slope ; it is of the form (s,s) which has length $\sqrt{s^2+s^2}=\sqrt{2}s$; thus, let $s=\frac{1}{\sqrt{2}}$. Unit Normal is $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

Problem 1 Continued.

Let E be the ellipse in the plane defined by the equation $\frac{x^2}{9}+\frac{y^2}{16}=1$ and let R be the set of points contained inside or on the ellipse. Here by $g(t)=(3\cos t, 4\sin t), 0\le t\le 2\pi$.

(e) Show that $|\ g'(t)\ |$ may be written as $\sqrt{9+7\cos^2 t}$

$$|\ g'(t)| = |\ (-3\sin t, 4\cos t)| = \sqrt{9\ sin^2 t + 16\ cos^2 t} = \sqrt{9\ sin^2 t + 9\ cos^2 t + 7\ cos^2 t}$$
$$= \sqrt{9+7cos^2 t}$$

(f) Find the curvature of the ellipse at the point corresponding to $t=\pi/3$.

The following may be helpful:

$$U(t) = \frac{\cos t}{(9+7\ cos^2 t)^{\frac{1}{2}}} \text{ has } U'(t) = \frac{-9\ \sin t}{(9+7\ cos^2\ t)^{\frac{3}{2}}}$$

$$V(t) = \frac{\sin t}{(9+7\ cos^2 t)^{\frac{1}{2}}} \text{ has } V'(t) = \frac{16\ \cos t}{(9+7\ cos^2 t)^{\frac{3}{2}}}$$

Unit tangent $T(t) = \frac{g\prime(t)}{|g'(t)|} = \left(\frac{-3sint}{\sqrt{9+7cos^2 t}}, \frac{4\cos t}{\sqrt{9+7cos^2 t}}\right)$.

Thus $T'(t) = \left(\frac{-48\cos t}{(9+7\ cos^2 t)^{\frac{3}{2}}}, \frac{-36\sin t}{(9+7\ cos^2 t)^{\frac{3}{2}}}\right) = 12\left(\frac{-4\cos t}{(9+7\ cos^2 t)^{\frac{3}{2}}}, \frac{-3\sin t}{(9+7\ cos^2 t)^{\frac{3}{2}}}\right)$

So $|T'(t)| = 12\sqrt{\frac{16\ cos^2 t + 9 sin^2 t}{(9+7cos^2 t)^3}} = 12\sqrt{\frac{9+7cos^2 t}{(9+7cos^2 t)^3}} = 12\sqrt{\frac{1}{(9+7cos^2 t)^2}} = \frac{12}{9+7cos^2 t}$.

Since $\cos\pi/3 = 1/2$, $\left|\ g'\left(\frac{\pi}{3}\right)\right| = \sqrt{9+7/4} = \sqrt{\frac{36+7}{4}} = \frac{\sqrt{43}}{2}$ and $|T'\left(\frac{\pi}{3}\right) = \frac{12}{\frac{43}{4}} = \frac{48}{43}$

$$\kappa\left(\frac{\pi}{3}\right) = \frac{|T'\left(\frac{\pi}{3}\right)|}{|\ g'\left(\frac{\pi}{3}\right)|} = \frac{48/43}{\sqrt{43}/2} = \frac{96}{43\sqrt{43}} = \frac{96\sqrt{43}}{43^2}$$

2. Again, Let E be the ellipse in the plane defined by the equation $\frac{x^2}{9} + \frac{y^2}{16} = 1$ and let R be the set of points contained inside or on the ellipse.

(a) Set up but do not evaluate an integral whose value would be the length of the ellipse.

$$\text{Length} = \int_0^{2\pi} |\, g'(t)|\, dt = \int_0^{2\pi} \sqrt{9 + 7cos^2 t}\, dt$$

(b) Find the area of R.

Method I: Area = 4 (Area in First Quadrant)

$= 4\int_{x=0}^{3} \int_{y=0}^{(\frac{4}{3})(\sqrt{9-x^2})} 1\, dy\, dx = 4\int_0^3 \frac{4}{3}\sqrt{9-x^2}\, dx = \frac{16}{3}\int_0^3 \sqrt{9-x^2}\, dx$

But this last integral is the area of a quarter of a circle of radius 3 which is $\frac{1}{4}\pi 3^2 = \frac{9}{4}\pi$.

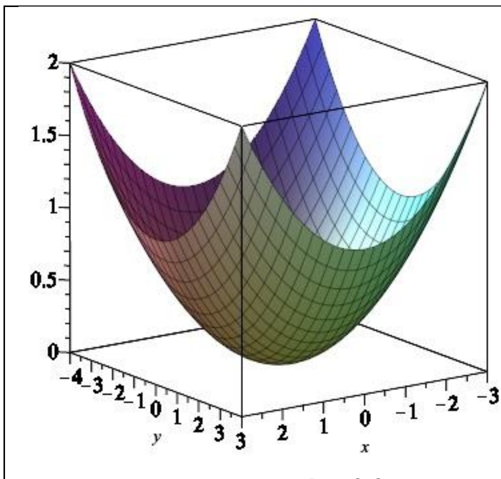
Hence area of the ellipse is $\frac{16}{3}\frac{9}{4}\pi = 12\pi$.

Method II: Let **F** be the vector field **F**(x, y) = ($-y$, x). Then scalar curl of **F** is (1 – (-1)) =2. By Green's Theorem

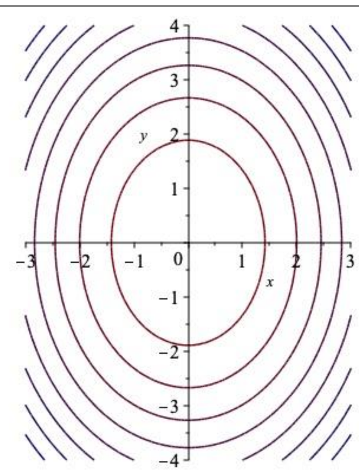
$$\int_E \mathrm{F} = \int_{\boldsymbol{R}} curl\, \mathbf{F} = \int_{\boldsymbol{R}} 2 = 2(Area\ of\ R).$$

Thus Area of $R = \frac{1}{2}\int_E \mathbf{F} = \frac{1}{2}\int_0^{2\pi} \mathbf{F}(g(t)) \cdot g'(t) dt = \frac{1}{2}\int_0^{2\pi}(-4\,\mathrm{sint}, 3\cos t) \cdot (-3\sin t, 4\cos t)\, dt = \frac{1}{2}\int_0^{2\pi} 12\, sin^2 t + 12 cos^2 t\, dt = \frac{1}{2}\int_0^{2\pi} 12\, dt = \frac{1}{2}(24\pi) = 12\pi.$

(c) Sketch a graph of the function $f(x,y) = \frac{x^2}{9} + \frac{y^2}{16}$ for $-3 \le x \le 3$, $-4 \le y \le 4$. Describe the level curves for this function.



Graph of f



Level Curves: $f(x,y) = C$ is an ellipse if C > 0, the point (0,0) if $C = 0$ and empty if C <0.

Problem 2 Continued.

Again, Let E be the ellipse in the plane defined by the equation $\frac{x^2}{9}+\frac{y^2}{16}=1$ and let R be the set of points contained inside or on the ellipse.

(d) Let **F** be the vector field $\mathbf{F}(x,y)=(x^2,xy)$. Find $\int_E \boldsymbol{F}$.

Method I: Use definition of the line integral and the parametrization $g(t)=(3\cos t, 4\sin t)$.

$$\int_E \mathbf{F} = \int_0^{2\pi} \mathbf{F}(g(t)\cdot g'(t)dt = \int_0^{2\pi}(9\,cos^2 t, 12\sin t\cos t)\cdot(-3\sin t, 4\cos t)\,dt =$$

$$\int_0^{2\pi} -27cos^2 t\sin t + 48\sin t\,cos^2 t\,dt = \int_0^{2\pi} 21\,cos^2 t\,sin\,t\,dt = -9\,cos^3 t_{t=0}^{t=2\pi} = 0$$

Method II: Use Green's Theorem: $\int_E \mathbf{F} = \int_{\boldsymbol{R}} \text{curl}\,\mathbf{F} = \int_{\boldsymbol{R}}(y-0) = \int_{\boldsymbol{R}} y = 0$ by symmetry of the region across the horizontal axis.

(e) Earlier in our course, we studied the temperature function $T(x,y)=2x^2+4y^2+2x+1$ defined on the unit disk. Find and classify all the critical points of T defined on the region R containing all the points inside or on the ellipse E. What are the warmest and coldest temperatures and where do they occur?

$\nabla T = (4x+2,8y)$ which is (0,0) only at the point $\left(-\frac{1}{2},0\right)$ in the interior of the disk where the temperature is ½. By completing the square in x, note that $T(x,y)=2\left(x+\frac{1}{2}\right)^2+4y^2+\frac{1}{2}$ which is clearly minimized at $\left(-\frac{1}{2},0\right)$.

We also need to check for extreme points on the ellipse E itself. We can use the method of Lagrange multipliers or simply note that on E, we have $y^2=16\left(1-\frac{x^2}{9}\right)$ so we can write the temperature as function of x alone:

$T(x)=2x^2+64\left(1-\frac{x^2}{9}\right)+2x+1$ which has $T'(x)=4x-\frac{128x}{9}+2, T''(x)=4-\frac{128}{9}<0$ so there is a relative maximum when $T'(x)=0$ which occurs at $x=\frac{9}{46}$ where T takes on its largest temperature 2999/46. The corresponding y values are $\pm\frac{14\sqrt{43}}{23}$

3. Let **F** be the vector field $\mathbf{F}(x, y, z) = \left(\cos x\, e^y z^3,\ \sin x\ \ e^y z^3,\ 3\sin x\, e^y z^2 - \frac{1}{z}\right)$

(a) Find the divergence of **F** at the point (π/2, 0, 1).

Div **F**(*x*,*y*,*z*) = $(\cos x\, e^y z^3)_x + (\sin x\ \ e^y z^3)_y + \left(3\sin x\, e^y z^2 - \frac{1}{z}\right)_z$

$= -\sin \boldsymbol{x}\ \ e^y z^3 + \quad \sin x\ \ e^y z^3 + 3\sin x\, e^y(2z) + \frac{1}{z^2} = 6 sin\ x\ e^y z + \frac{1}{z^2}$

so Div **F**(π/2, 0, 1) = 6 sin $\left(\frac{\pi}{2}\right) e^0\ \times 1 + \frac{1}{1} = 6 + 1 = 7.$

(b) Find the curl of **F**.

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \boldsymbol{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x\, e^y z^3 & \sin x\ e^y z^3 & 3\sin x\, e^y z^2 - \frac{1}{z} \end{pmatrix}$$

$= (3\sin x\, e^y z^2 - 3\sin x\ \ e^y z^2)\mathbf{i} - (3\cos x\, e^y z^2 -\ 3\cos x\, e^y z^2)\mathrm{j} +$
$(\cos x\ \ e^y z^3 - \cos x\, e^y z^3)\ \boldsymbol{k}$ = (0, 0, 0)**.**

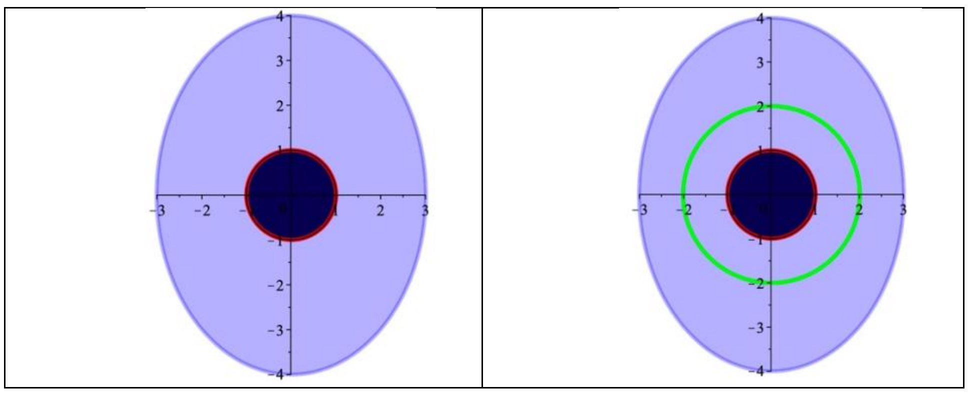
(c) Show that **F** is a conservative vector field.
Method I: The vector field **F** is defined on all of 3-space which is simply connected. Since the curl is identically 0,we have a theorem that says **F** must be conservative.
Method II: Find the potential function as in part (d).

(d) Find a potential function for this vector field **F**.
First integrate the first component of **F** with respect to *x* which gives us a first candidate $f(x, y, z) = \sin x\, e^y z^3 + G(y, z)$.Then
$f_y(x, y, z) = \ \sin x\, e^y z^3 + G_y(y, z)$ which matches up with the second component of the vector field if $G_y(y, z) = 0$ which is true if *G* is purely a function *H(z)* of *z*.
Thus $f(x, y, z) = \sin x\, e^y z^3 + H(z)$ and the derivative with respect to *z* is
$3\sin x\, e^y z^2 + H'(z)$. Hence we need $H'(z) = -\frac{1}{z}$ so we can choose $H(x) = \ -\ln z$.

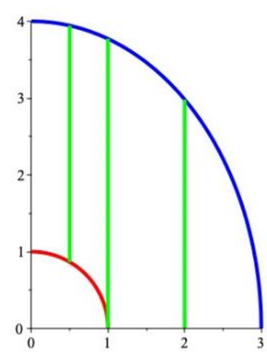
A potential function is $f(x, y, z) = \sin x\, e^y z^3 - \ln z$

4. Let S be the region between the unit circle C ($x^2 + y^2 = 1$) and our ellipse E ($\frac{x^2}{9} + \frac{y^2}{16} = 1$).

(a) Is S simply connected? Explain.
S is the blue shaded region shown on the left below. It does not contain the black hole, which is the interior of the unit disk. The green loop on the right (the circle of radius 2 with center at the origin) can not be shrunk to a point inside S so S is **not** simply connected.



(b) Let Q be the portion of the region S that lies in the first quadrant. Write the area of Q as a sum of double integrals in Cartesian coordinates. Do **not** evaluate the integrals.
Slice Q up into vertical slices.



For $0 \le x \le 1$,the slices run up from the circle to the ellipse, but for $0 \le x \le 3$, the start at the horizontal axis and run up to the ellipse.W e will need two integrals:

$$\int_{x=0}^{x=1} \int_{y=\sqrt{1-x^2}}^{y=\frac{4}{3}\sqrt{9-x^2}} 1\, dy\, dx + \int_{x=1}^{x=3} \int_{y=0}^{y=\frac{4}{3}\sqrt{9-x^2}} 1\, dy\, dx$$

Write the area of Q as a sum of double integrals in polar coordinates. Do not evaluate the integrals. [Hint: Show the polar equation for the ellipse E has the form $r = \frac{12}{\sqrt{16\cos^2\theta + 9\sin^2\theta}}$ for points in Q.].

Rewrite $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as $b^2x^2 + a^2y^2 = a^2b^2$ and let $x = r\cos\theta, y = r\sin\theta$.
Then the equation takes the form $r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$ so

$$r = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$Area = \int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=\frac{12}{\sqrt{16\cos^2\theta + 9\sin^2\theta}}} r\, dr\, d\theta$$

5. Let γ be the curve in 3-space parametrized by $\boldsymbol{g}(t) = (\sin t, \cos t, \sin t - \cos t), 0 \le t \le 2\pi$.

Verify Stokes' Theorem for γ and the vector field $\mathbf{F}(x, y, z) = (yz, xz, xy)$

Note: This is Problem 43 of Chapter 8 which was Assignment 35.

The curl of the vector field **F** is

$\left((xy)_y - (xz)_z, (yz)_z - (xy)_x, (xz)_x - (yz)_y\right) = (x - x, y - y, z - z) = (0,0,0)$

Hence $\int_S curl\, \mathbf{F} = \int_S \mathbf{0} = 0$.

Now $\boldsymbol{g}(t) = (\sin t, \cos t, \sin t - \cos t)$ gives $\boldsymbol{g}'(t) = (\cos t, -\sin t, \cos t + \sin t)$

And $\mathbf{F}(\boldsymbol{g}(t))$ =$(\sin t \cos t - cos^2 t,\ sin^2 t - \sin t \cos t, \sin t \cos t)$.

Then

$$\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{t})) \cdot \boldsymbol{g}'(\boldsymbol{t}) = 2 \sin t\, cos^2 t + 2\, sin^2 t \cos t - sin^3 t - cos^3 t$$

The line integral of **F** around the boundary of *S* is the line integral over the curve γ with parametrization **g:**

$$\int_{\partial S} \mathbf{F} = \int_0^{2\pi} 2 \sin t\, cos^2 t + 2\, sin^2 t \cos t - sin^3 t - cos^3 t\ dt$$

But each of these terms has an integral of 0, so $\int_{\partial S} \mathbf{F} = 0 = \int_S curl\, \mathbf{F}$

Evaluating the line integral is much simpler if we note that **F** is the gradient of the potential function *f(x,y,z) = xyz.*