1. A relation on a set $A$ is a binary predicate $R$ that for each $a, b \in A$ assigns either true or false to the proposition "$a R b$.

(a) A relation can simply be considered to be a binary predicate.

[Though the traditional formal definition of a relation in mathematics is as the set of ordered pairs $(a, b)$ for which $a R b$ is true, which is not the way you should conceptualize a relation.]

(b) Some relations (predicates!) we’ve already seen are:

(i) on $\mathbb{Z}$: $=, \neq, <, >, \leq, \geq$ (reflexive, symmetric, transitive)

(ii) on the set of vertices of a graph $G$:

- $v_0$ is joined by an edge to $v_1$ (symmetric)
- $v_0$ is joined by a path to $v_1$ (reflexive, symmetric, transitive :: equivalence relation)

(iii) on the set of subgraphs of a graph $G$:

- $H$ is a subgraph of $K$ (reflexive, transitive)
- $H$ is a proper subgraph of $K$ (transitive)
- $H$ is disjoint from $K$ (symmetric)

2. A relation $R$ on a set $A$ is...

(a) reflexive just when $\forall a \in A$, $a R a$ (e.g., "is joined by a path to", "is a subgraph of")

(b) symmetric just when $\forall a, b \in A$, $a R b \Leftrightarrow b R a$ (e.g., "is joined by an edge/path to", "is disjoint from")

(c) transitive just when $\forall a, b, c \in A$, $a R b \land b R c \Rightarrow a R c.$ (e.g., "is joined by a path to", "is a [proper] subgraph of")
3. (a) An **equivalence relation** on a set $A$ is one that is reflexive, symmetric, and transitive.

(b) For each $a \in A$, we can form the **equivalence class** of $a$ as:

$$[a] = \{ b : a \mathrel{R} b \}$$

which is also sometimes written $\overline{a}$ or $a$

(c) These collection $\{ [a] : a \in A \}$ of all equivalence classes form a **partition** of the set $A$ because

1. Each one is nonempty,
2. Distinct equivalence classes are disjoint,
3. Their union is the set $A$.

(They break the set $A$ up into pieces, much like a jigsaw puzzle)

If our equivalence relation is \( \sim \), we express this collection of equivalence classes by \( A/\sim \).

(d) Any given relation can be "grown" into the minimal equivalence relation subsuming it (i.e., such that every pair related by the original relation is still related by the equivalence relation).

Formally, given any relation $R$ on $A$, we can define such an equivalence relation $\sim$ by:

$$x \sim y \iff \exists \text{ an equivalence relation } S \text{ subsuming } R, xS y.$$
4. With the relation $|n|$ on $\mathbb{Z}$ defined by $a \mid b \iff \exists k \in \mathbb{Z}$ with $b = k \cdot a$:

(since $a \nmid b \iff \forall k \in \mathbb{Z}, \ b \neq k \cdot a$)

Claim: $4 \mid 12$, i.e., $\exists k \in \mathbb{Z}$ with $12 = k \cdot 4$

Proof: Take $k = 3 \in \mathbb{Z}$ then $1 \cdot 4 = 3 \cdot 4 = 12$, so $4 \mid 12$.

Claim: $6 \mid 12$, i.e., $\exists k \in \mathbb{Z}$ with $12 = k \cdot 2$

Proof: Take $k = 2 \in \mathbb{Z}$ then $2 \cdot 6 = 2 \cdot 6 = 12$, so $6 \mid 12$.

Claim: $12 \mid 12$, i.e., $\exists k \in \mathbb{Z}$ with $12 = k \cdot 12$

Proof: Take $k = 1 \in \mathbb{Z}$ then $1 \cdot 12 = 1 \cdot 12 = 12$, so $12 \mid 12$.

Claim: $24 \mid 12$, i.e., $\forall k \in \mathbb{Z}, 12 \neq k \cdot 24$

Proof: Let $k \in \mathbb{Z}$ be given, and suppose (for contradiction) that $12 = k \cdot 24$.

This would mean that $\frac{1}{2} = k$, contradicting $k \in \mathbb{Z}$.

Claim: $5 \mid 12$, i.e., $\forall k \in \mathbb{Z}, 12 \neq k \cdot 5$

Proof: Let $k \in \mathbb{Z}$ be given, and suppose (for contradiction) that $12 = k \cdot 5$.

This would mean that $\frac{12}{5} = k$, contradicting $k \in \mathbb{Z}$.

6. This relation is both reflexive and transitive. (Try to write out & prove these!)
5. Fix a positive integer \( n \), and define the relation on \( \mathbb{Z} \):
\[
a \equiv b \pmod{n} \quad \text{just when } \quad n \mid (b-a).
\]

(a) Claim: \( 6 \equiv 1 \pmod{5} \), i.e., \( 5 \mid (1-6) \), i.e., \( 5 \mid -5 \), i.e., \( \exists k \in \mathbb{Z} \) with \(-5 = k \cdot 5\).

Proof: Take \( k = -1 \in \mathbb{Z} \). Then \( k \cdot 5 = (-1) \cdot 5 = -5 \), so \( 5 \mid -5 \), and thus \( 6 \equiv 1 \pmod{5} \). \( \blacksquare \)

Claim: \( 4 \not\equiv 1 \pmod{5} \), i.e., \( 5 \nmid (1-4) \), i.e., \( 5 \nmid -3 \), i.e., \( \forall k \in \mathbb{Z}, -3 \not\equiv k \cdot 5 \).

Proof: Let \( k \in \mathbb{Z} \) be given, and suppose (for contradiction) that \(-3 \equiv k \cdot 5\).

This would mean that \( -\frac{3}{5} = k \), contradicting \( k \in \mathbb{Z} \).

So \( 4 \not\equiv 1 \pmod{5} \). \( \blacksquare \)

(b) To show that \( \equiv \pmod{n} \) is an equivalence relation on \( \mathbb{Z} \), we must show reflexivity, symmetry, and transitivity:

Claim: \( \forall a \in \mathbb{Z}, \ a \equiv a \pmod{n} \) i.e., \( n \mid (a-a) \), or \( n \mid 0 \), i.e., \( \exists k \in \mathbb{Z} \) with \( 0 = k \cdot n \)

Proof: Let \( a \in \mathbb{Z} \) be given.

Take \( k = 0 \in \mathbb{Z} \). Then \( k \cdot n = 0 \cdot n = 0 \), so \( n \mid 0 \). \( \blacksquare \)

Claim: \( \forall a, b \in \mathbb{Z}, \ a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n} \)

Proof: Let \( a, b \in \mathbb{Z} \) be given, and suppose \( a \equiv b \pmod{n} \), i.e., \( n \mid (b-a) \), i.e., \( \exists k \in \mathbb{Z} \) with \( -a = k \cdot n \).

Taking such a \( k \in \mathbb{Z} \), we have \( b-a = kn \). \( \blacksquare \)

Take \( l = -k \in \mathbb{Z} \). Then \( kn = (-k)n = -(k \cdot n) = -(b-a) \) by (x), so \( a-b = kn \), i.e., \( a \equiv b \pmod{n} \).

\( \blacksquare \)
**Claim:** \(\forall a, b, c \in \mathbb{Z}, \ a \equiv b \pmod{n} \land b \equiv c \pmod{n} \implies a \equiv c \pmod{n}\)

**Proof:** Let \(a, b, c \in \mathbb{Z}\) be given, and suppose:

- \(a \equiv b \pmod{n}\), i.e., \(n \mid (b-a)\), i.e. \(\exists k \in \mathbb{Z}\) with \(b-a = kn\)
- \(b \equiv c \pmod{n}\), i.e., \(n \mid (c-b)\), i.e. \(\exists l \in \mathbb{Z}\) with \(c-b = ln\)

Taking such \(k, l \in \mathbb{Z}\), we have \(b-a = kn \quad \text{(m)}\)

\[b-a = kn \quad \text{(m)}\]

\[c-b = ln \quad \text{(m)}\]

Take \(w = k + l \in \mathbb{Z}\).

Then \(wn = (k+l)n = kn + ln\)

\[= (b-a) + (c-b)\]

\[= b-a + c-b\]

\[= c-a. \quad \blacksquare\]

**Equivalence Classes**

For \(n = 1 \land a \in \mathbb{Z}\), \(\bar{a} = \{ b \in \mathbb{Z} : a \equiv b \pmod{1} \}\)

\[= \{ b \in \mathbb{Z} : 1 \mid (b-a) \}\]

But \(1 \mid \) every integer, so \(\forall b \in \mathbb{Z}, b \in \bar{a}\)

\[\therefore \bar{a} = \mathbb{Z}\]

just one equivalence class, of \(\mathbb{Z}\)!

For \(n = 2 \land a \in \mathbb{Z}\), \(\bar{a} = \{ b \in \mathbb{Z} : a \equiv b \pmod{2} \}\)

\[= \{ b \in \mathbb{Z} : 2 \mid (b-a) \}\]

In other words, \(b \in \bar{a}\) just when \(b-a\) is even, or \(b = a + an\) an even number, so

\[\bar{a} = \{ \ldots, a-4, a-2, a, a+2, a+4, a+6, \ldots \}\]

(1) If \(a\) is even: \(\{ \ldots, -6, -4, -2, 0, 2, 4, 6, \ldots \} = \bar{0}\)

(2) If \(a\) is odd: \(\{ \ldots, -5, -3, -1, 1, 3, 5, \ldots \} = \bar{1}\)

These are the two equivalence classes.

Note that every integer is in exactly one of these two sets!
For \( n=5 \) and \( a \in \mathbb{Z} \), the equivalence classes similarly consist of integers differing by multiples of 5:

\[
[a] = \{ ..., a-10, a-5, a, a+5, a+10, ... \}
\]

There are only five cases, so five equivalence classes:

\[
\begin{align*}
\{ ..., -10, -5, 0, 5, 10, ... \} &= \overline{0} \\
\{ ..., -9, -4, 1, 6, 11, ... \} &= \overline{1} \\
\{ ..., -8, -3, 2, 7, 12, ... \} &= \overline{2} \\
\{ ..., -7, -2, 3, 8, 13, ... \} &= \overline{3} \\
\{ ..., -6, -1, 4, 9, 14, ... \} &= \overline{4}
\end{align*}
\]

And \( \overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4} \) are exactly the five equivalence classes.

In general, the equivalence classes \( \mod n \) will be

\[
\{ \overline{0}, \overline{1}, \overline{2}, ..., \overline{n-1} \}
\]

(Dividing any \( a \in \mathbb{Z} \) by \( n \) leaves exactly one of 0, 1, 2, ..., \( n-1 \) as the remainder, so \( a \) and that remainder differ by a multiple of \( n \) \( \therefore \) are in the same equivalence class.)

NOTE THAT EVERY INTEGER IS IN EXACTLY ONE OF THESE FIVE SETS!
6. Fix a positive integer \( n \), and suppose \( a, a', b, b' \in \mathbb{Z} \) satisfy:

\[ a \equiv a' \pmod{n}, \text{ i.e., } n \mid (a' - a), \text{ i.e., } \exists k \in \mathbb{Z} \text{ with } a' - a = kn \tag{\star} \]

\[ b \equiv b' \pmod{n}, \text{ i.e., } n \mid (b' - b), \text{ i.e., } \exists l \in \mathbb{Z} \text{ with } b' - b = ln \tag{\star\star} \]

(a) **Claim:** \( a + b \equiv a' + b' \pmod{n} \), i.e., \( n \mid [(a' + b') - (a + b)] \)

**Proof:**

\[ (a' + b') - (a + b) = (a' - a) + (b' - b) = kn + ln = (k + l)n. \]

Since \( k + l \in \mathbb{Z} \), \( n \mid (k + l)n \), so \( n \mid [(a' + b') - (a + b)] \), and thus \( a + b \equiv a' + b' \pmod{n} \). \[\square\]

(b) **Claim:** \( a - b \equiv a' - b' \pmod{n} \), i.e., \( n \mid [(a' - b') - (a - b)] \)

**Proof:**

\[ (a' - b') - (a - b) = (a' - a) - (b' - b) = kn - ln = (k - l)n. \]

Since \( k - l \in \mathbb{Z} \), \( n \mid (k - l)n \), so \( n \mid [(a' - b') - (a - b)] \), and thus \( a - b \equiv a' - b' \pmod{n} \). \[\square\]

(c) **Claim:** \( a \cdot b \equiv a' \cdot b' \pmod{n} \), i.e., \( n \mid (a' \cdot b' - a \cdot b) \)

**Proof:**

\[ a' \cdot b' - a \cdot b = a' \cdot b' - a \cdot b' + a \cdot b' - a \cdot b \]

\[ = (a' - a) b' + a (b' - b) \]

\[ = (kn) b' + a (ln) \quad \text{by } \star \text{ and } \star\star \]

\[ = (k b' - a l) n. \]

Since \( k b' - a l \in \mathbb{Z} \), \( n \mid (k b' - a l) n \), so \( n \mid (a' \cdot b' - a \cdot b) \), and thus \( a' \cdot b' \equiv a \cdot b \pmod{n} \). \[\square\]
7. \( G \): Simple graph, with two relations on its set of vertices:

- \( V_0 \sim A \sim V_1 \) if there is an edge joining \( V_0 \) to \( V_1 \) (adjacent vertices)
- \( V_0 \sim C \sim V_1 \) if there is a path joining \( V_0 \) to \( V_1 \) (vertices in the same component)

\( C \) is an equivalence relation (easy to check: reflexivity, symmetry, & transitivity), but \( A \) is not (it's only symmetric).

Two vertices are equivalent under \( C \), just when they are connected by a path, meaning that they are in the same component (why? the path connecting them is nonempty & connected, and thus is contained in some maximal nonempty connected subgraph — i.e., component — of \( G \).

:: the equivalence classes for \( C \) just consist of the vertices in each component of \( G \)

\( G \) is the equivalence relation "generated by" \( A \), because transitivity forces all vertices in a chain of \( A \) — i.e., a path — to be equivalent in any equivalence relation subsumed \( A \).

8. \( G \): Digraph; for vertices \( V_0, V_1 \) of \( G \),

- \( V_0 \sim SC \sim V_1 \) means \( \exists \) directed path (w \( G \)) from \( V_0 \) to \( V_1 \)
  and \( \exists \) directed path (w \( G \)) from \( V_1 \) to \( V_0 \)

(a) To show that \( SC \) is an equivalence relation on the vertices of \( G \),
we need to show reflexivity, symmetry, and transitivity:

Claim: \( V \) vertices \( a, b \) of \( G \), \( a \sim SC \sim b \)

\* Need directed paths from \( a \) to \( b \) and from \( b \) to \( a \)

Proof: Let a vertex \( a \) of \( G \) be given.

The constant path of length 0 at \( a \) connects \( a \) to \( a \).

Claim: \( V \) vertices \( a, b \) of \( G \), \( a \sim SC \sim b \)

\* Need directed paths from \( a \) to \( b \) and from \( b \) to \( a \)

Proof: Let vertices \( a, b \) of \( G \) be given.

And suppose that \( a \sim SC \sim b \), i.e., \( \exists \) directed paths \( p \) from \( a \) to \( b \)
and \( q \) from \( b \) to \( a \).

Then \( b \) is a directed path from \( b \) to \( a \), and \( p \) is a directed path from \( a \) to \( b \).
So by definition, \( b \sim SC \sim a \).
Claim: ∀ vertices \( a, b, c \) of \( G \), \( a \triangleleft b \land b \triangleleft c \Rightarrow a \triangleleft c \).

Proof: Let vertices \( a, b, c \) of \( G \) be given,

and suppose \( a \triangleleft b \), i.e., \( \exists \) directed paths \( p \) from \( a \) to \( b \) and \( q \) from \( b \) to \( a \).

And \( b \triangleleft c \), i.e., \( \exists \) directed paths \( r \) from \( b \) to \( c \) and \( s \) from \( c \) to \( b \).

Then concatenating\(^*\) the paths \( p \& q \) yields a directed path from \( a \) to \( c \),
and concatenating\(^*\) the paths \( s \& r \) yields a directed path from \( c \) to \( a \).

So by definition, \( a \triangleleft c \).\(^*\)

\(^*\)Technically, the paths we’re concatenating could share some vertices other than their endpoints, but this can be easily fixed: let \( v \) be the first vertex allowed \( p \) that it shares with \( r \), and trim off everything after \( v \) in \( p \) and everything before \( v \) in \( r \) first. (Similarly for the paths \( s \& p \).)

(c) (Picture each red blob as a big vertex to see \( G \)!) \( G \) is clearly a directed graph — with must it be acyclic? (Abuse of term?!) What about \( G \)?

Suppose (for contradiction) that \( G \) had a directed cycle:

This would give directed paths in both directions between vertices of \( G \) in different equivalence classes, making them equivalent (contradiction!)!