1. (a) CLAIM: For any graph G, if G has a loop or parallel edges, then G is not acyclic.

PROOF: Let a graph G be given, and suppose G has a loop \( \bigcirc = \) 1-cycle in G.

\[ \text{Q.E.D.} \]

\( \square \) G has parallel edges \( \bigcirc = \) 2-cycle in G.

In either case, G contains a cycle, so G is not acyclic. \( \blacksquare \)

(b) For any graph G, if G is acyclic, then G has no loops or parallel edges.

\[ \text{Q.E.D.} \]

(c) 1 & 2-cycles give loops & parallel edges, as in (a), so the minimum number of vertices in a simple cycle is 3.

(d) CLAIM: For every simple cycle \( C \), there exist vertices \( v_0, v_1 \) of \( C \) with two distinct paths connecting them.

PROOF: Let a simple cycle \( C \) be given.

By part (c), \( C \) has at least 3 vertices.

Take two adjacent ones, \( v_0 \) & \( v_1 \).

Then (1) the edge between them and (2) the rest of the cycle give two distinct paths connecting them. \( \blacksquare \)

2. (a) CLAIM: Every tree \( T \) is a simple graph.

i.e. \( \forall n \geq 1, \) every tree \( T \) with \( n \) vertices is simple.

PROOF: \( P(1) \): Let a tree \( T \) with 1 vertex be given.

By definition of a tree, \( T \) consists of one isolated vertex. This is simple because it has no edges.

\[ \forall n \geq 1, \] \( P(n) \rightarrow P(n+1) \): \( \rightarrow \) \( \forall n \geq 1, \) every tree \( T \) with \( n+1 \) vertices is simple.

Let \( n \geq 1 \) be given, and suppose \( P(n) \), i.e., every tree \( T \) with \( n \) vertices is simple (x).

Let a tree \( T' \) with \( n+1 \) vertices be given.

By definition of a tree, \( T' \) is built from some tree \( T \) with \( n \) vertices by adding a leaf & pendant edge.

By (x), \( T \) is simple, i.e., \( T \) has no loops or parallel edges.

The only new edge in \( T' \) is a pendant edge to a new vertex, so it is neither a loop nor parallel to another edge.

Thus \( T' \) is also simple. \( \blacksquare \)
(6) **Claim**: For every tree $T$ and each pair of vertices $v_0, v_1$, there is a unique path connecting $v_0$ and $v_1$.

Proof: **$P(1)$**: Let a tree $T$ with 1 vertex be given, and let vertices $v_0, v_1$ of $T$ be given. By the definition of a tree, $T$ consists of one isolated vertex, so $v_0 = v_1$, and there are no edges, so there is just one path (of length 0) connecting $v_0$ and $v_1$. 

Let $n \geq 1$ be given, and suppose $P(n)$ (see $P(n)$ above).

Let a tree $T'$ with $(n+1)$ vertices be given, and let vertices $v_0, v_1$ of $T'$ be given.

By the definition of a tree, $T'$ is built from some tree $T$ with $n$ vertices by adding a leaf and a pendant edge.

- If neither $v_0$ nor $v_1$ is this new leaf, then by (*) there is a unique path in $T$ connecting $v_0$ and $v_1$. Any path involving our new leaf must start or end there, so the path in $T$ is the unique path in $T'$ connecting $v_0$ and $v_1$.

- If $v_0' = v_1'$ is the new leaf above, there is a unique path of length 0 connecting them.

- Otherwise, say $v_0' \in T$ and $v_1'$ is the new leaf.

By ($\ast$), there is a unique path in $T$ connecting $v_0'$ and the vertex $x$ adjacent to the leaf. Adding the pendant edge and the new leaf gives the unique path in $T'$ connecting $v_0'$ to the leaf $v_1'$.
(c) Claim: If T is a tree, then $n(T) = e(T) + 1$

I.e., $\forall n \geq 1$, if T is a tree with n vertices, then $n(T) = e(T) + 1$.

Proof. P(1): Suppose that T is a tree with 1 vertex.

Then T consists of one isolated vertex & no edges.

Thus $n(T) = 1$ & $e(T) = 0$, so $n(T) = e(T) + 1$.

Let $n \geq 1$ be given,

and suppose P(n), i.e., if T is a tree with n vertices, then $n(T) = e(T) + 1$ (1)

Suppose that $T'$ is a tree with $(n+1)$ vertices.

By definition of a tree, $T'$ is built from a tree T with n vertices by adding a loop edge and a pendant edge.

Thus, $n(T') = n(T) + 1$ and $e(T') = e(T) + 1$.

And by (1), $n(T) = e(T) + 1$,

so $n(T') = n(T) + 1 = [e(T) + 1] + 1 = e(T') + 1$. \(\blacksquare\)
3. (a) **Every tree is nonempty**, directly from the definition of a tree: we start with one vertex and add pairs of vertices & edges to it.

   (b) **Claim**: **Every tree** $T$ **is acyclic**

   **Proof**: let a tree $T$ be given,

   and suppose (for contradiction!) that $T$ contains a cycle $C$.

   By 1(d), there exist vertices $v_0, v_1$ of $C$ (and thus of $T$) with two distinct paths in $C$ (and thus in $T$) connecting them.

   This contradicts 2(b): in a tree, there must be exactly one path connecting them.

   (c) **Every tree** $T$ **is connected** follows immediately from 2(b),

   which is just the definition of "connected" with the extra condition of uniqueness.

   (d) **Claim**: if $G$ has a spanning tree $T$, then $G$ is connected $v_0, v_1$.

   **Proof**: suppose that $G$ has a spanning tree $T$.

   let vertices $v_0, v_1$ of $G$ be given;

   because $T$ is a spanning tree for $G$, $v_0, v_1$ are vertices of $T$.

   so by 2(b), there is a unique path in $T$ connecting $v_0, v_1$.

   but $T$ is a subgraph of $G$, so this is a path in $G$ connecting $v_0, v_1$. ■
3. (a) **Every tree is nonempty, directly from the definition of a tree:**
   We start with one vertex and add pairs of vertices & edges to it.

   (b) **Claim:** **Every tree** $T$ **is acyclic**

   **Proof:** Let a tree $T$ be given,
   and suppose (for contradiction!) that $T$ contains a cycle $C$.
   By (1)(a), there exist vertices $v_0, v_1$ of $C$ (and thus of $T$)
   with two distinct paths in $C$ (and thus in $T$) connecting them.
   This contradicts (2)(6): in a tree, there must be exactly
   one path connecting them.

   (c) **Every tree** $T$ **is connected** follows immediately from (2)(6),
   which is just the definition of "connected" with the extra
   condition of uniqueness.

   (d) **Claim:** **If $G$ has a spanning tree** $T$, **then** $G$ **is connected** $v_0, v_1$.

   **Proof:** Suppose that $G$ has a spanning tree $T$.
   Let vertices $v_0, v_1$ of $G$ be given;
   because $T$ is a spanning tree for $G$, $v_0, v_1$ are vertices of $T$.
   So by (2)(6), there is a unique path in $T$ connecting $v_0, v_1$.
   But $T$ is a subgraph of $G$, so this is a path in $G$ connecting $v_0, v_1$. 

   For each pair of vertices $v_i, v_j$ of $G$, in $G$!

   There is a path connecting $v_i, v_j$. 

4. **Claim:** If \( G \) is nonempty & connected, and if \( G \) is not a tree, then \( G \) contains a cycle.

**Proof:** Suppose that \( G \) is nonempty & connected, and that \( G \) is not a tree.

Because \( G \) is nonempty & connected, we can find a spanning tree \( T \) for \( G \). \( T \) is a subgraph of \( G \) and contains all vertices of \( T \), but \( G \) is not a tree (\( G \neq T \)), so \( G \) contains at least one edge not in \( T \).

Call this edge \( e \), and say it connects \( v_0 \) to \( v_1 \).

- If \( v_0 = v_1 \), we have a 1-cycle in \( G \).
- If \( v_0 \neq v_1 \), by 2(b), there is a path in \( T \) connecting \( v_0 \) & \( v_1 \); adding \( e \) to this path forms a cycle in \( G \).

In either case, \( G \) contains a cycle.

VIA THE CONTRAPPOSITIVE: If \( G \) is nonempty, connected, & acyclic, then \( G \) is a tree.

(In light of 3(a,b,c), \( G \) is a tree \( \implies G \) is nonempty, connected, & acyclic)

5. **Suppose that we add one edge to a tree \( T \), obtaining a graph \( G \).**

(a) By 2(c), \( v(T) = e(T) + 1 \) (because \( T \) is a tree). Adding one edge gives \( v(G) = v(T) \) and \( e(G) = e(T) + 1 \), so \( v(G) = v(T) = e(T) + 1 = e(G) = e(G) + 1 \).

By the contrapositive to 2(c), \( G \) is not a tree.

But \( G \) is connected and nonempty (because \( T \) was, as a tree), so by Problem 4, \( G \) has a cycle.

(b) Removing any edge of this cycle to get a new graph \( G' \) gives us a tree again (because it will be nonempty & connected but will now have \( v(G') = e(G') + 1 \) again).