NOTE THAT THE FIRST VALUE OF n IN A PROOF BY INDUCTION ON n COULD BE ANYTHING! (THIS ONLY CHANGES OUR FIRST STEP AND THE A IN OUR SECOND STEP.)

1. To prove \( \forall n \geq 0, 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}(2n^3 + 3n^2 + n) \) by induction on n:

**Claim:** \( P(0) \)

**Proof:** LHS of \( P(0) = 0 \) (no terms to add), and

RHS of \( P(0) = \frac{1}{6}(2 \cdot 0^3 + 3 \cdot 0^2 + 0) = 0 \), so LHS = RHS \( \checkmark \)

**Claim:** \( \forall n \geq 0, P(n) \Rightarrow P(n+1) \)

**Proof:** Let \( n \geq 0 \) be given.

Suppose \( P(n) \), i.e., \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}(2n^3 + 3n^2 + n) \) \( \quad (\star) \)

Then \( (1^2 + 2^2 + 3^2 + \cdots + n^2) + (n+1)^2 = \frac{1}{6}(2n^3 + 3n^2 + n) + (n+1)^2 \) by \( \star \)

\[
\frac{1}{6}(2n^3 + 3n^2 + n + 6[n^2 + 2n + 1]) = \frac{1}{6}(2n^3 + 9n^2 + 13n + 6)
\]

And \( \frac{1}{6} \left(2[n+1]^3 + 3[n+1]^2 + [n+1]\right) = \frac{1}{6}(2[n+1]^3 + 3[n+1]^2 + 3[n^2 + 2n + 1] + n+1) = \frac{1}{6} \left(2[n+1]^3 + 9[n+1]^2 + 13[n+1] + 6\right) \)

So LHS = RHS in \( P(n+1) \) \( \checkmark \)
2. Suppose that $n \neq 1$.

To prove that $\forall n \in \mathbb{N}, 1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$ by induction on $n$:

Claim: $P(0)$

Proof: LHS of $P(0) = 1$ (since $r^0 = 1$, so the sum has just one term), and
RHS of $P(0) = \frac{1 - r^{0+1}}{1 - r} = \frac{1 - r}{1 - r} = 1$, so LHS = RHS.

Claim: $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$

Proof: Let $n \in \mathbb{N}$ be given.

Suppose $P(n)$, i.e., $1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$ (x)

Then $1 + r + r^2 + \ldots + r^n + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + \frac{1 - r^{n+2}}{1 - r}$

By (x)

$= \frac{-r^{n+2}}{1 - r} = \text{RHS}$

3. (a) To prove $\forall n \geq 10, 100n < 2^n$ by induction on $n$:

Claim: $P(10)$

Proof: LHS of $P(10) = 100 \cdot 10 = 1000$, and
RHS of $P(10) = 2^{10} = 1024$, so LHS < RHS.

Claim: $\forall n \geq 10, P(n) \implies P(n+1)$

Proof: Let $n \geq 10$ be given.

Suppose $P(n)$, i.e., $100n < 2^n$ (x)

$100(n+1) = (100n) + 100 < 2^n + 100$ by (x)

By (x)

$< 2^n + 2^n \text{ since } n \geq 10$

$= 2 \cdot 2^n = 2^{n+1}$, i.e., $100(n+1) < 2^{n+1} \blacksquare$
(b) To prove $\forall n \geq 4, \quad \frac{2^n}{P(n)} \leq n!$ by induction on $n$:

**Claim:** $P(4)$

**Proof:** LHS of $P(4) = 2^4 = 16$, and
RHS of $P(4) = 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, so LHS < RHS.

**Claim:** $\forall n \geq 4, \quad (P(n) \Rightarrow P(n+1)) \Rightarrow 2^{n+1} < (n+1)!$

**Proof:** Let $n \geq 4$ be given.

**Suppose** $P(n)$, i.e., $2^n < n!$ (x)

Then $2^{n+1} = 2 \cdot 2^n < 2 \cdot n!$.

AND

$(n+1)! = (1 \cdot 2 \cdot \ldots \cdot n)(n+1) = n!(n+1)$,

So LHS < RHS because $2 < (n+1)$. ⚫

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(c) Suppose $x > -1$.

To prove $\forall n \geq 0, \quad (1+x)^n \geq 1+nx$ by induction on $n$:

**Claim:** $P(0)$

**Proof:** LHS of $P(0) = (1+x)^0 = 1$, and
RHS of $P(0) = 1 + 0 \cdot x = 1$, so LHS ≤ RHS.

**Claim:** $\forall n \geq 0, \quad (P(n) \Rightarrow P(n+1)) \Rightarrow (1+x)^{n+1} \geq 1+(n+1)x$

**Proof:** Let $n \geq 0$ be given.

**Suppose** $P(n)$, i.e., $(1+x)^n \geq 1+nx$ (x)

Then $(1+x)^{n+1} = (1+x)((1+x)^n) \geq (1+x)(1+nx)$, by (x)

which equals $1 + x + nx + nx^2 = 1 + (n+1)x + nx^2$

$\geq 1 + (n+1)x$ since $nx^2 \geq 0$ ⚫

RHS
4. \(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55\)
\[
\begin{align*}
F_0 & \downarrow \quad F_1 \downarrow \quad F_2 \downarrow \quad F_3 \downarrow \quad F_4 \downarrow \quad F_5 \downarrow \quad F_6 \downarrow \quad F_7 \downarrow \quad F_8 \downarrow \quad F_9 \downarrow \quad F_{10} \quad (F_{n+1} = F_n + F_{n-1}) \\
F_2 &= F_1 + F_0 \\
F_3 &= F_2 + F_1 \\
F_4 &= F_3 + F_2 \\
F_5 &= F_4 + F_3 \\
F_6 &= F_5 + F_4 \\
F_7 &= F_6 + F_5 \\
F_8 &= F_7 + F_6 \\
F_9 &= F_8 + F_7 \\
F_{10} &= F_9 + F_8
\end{align*}
\]

5. (a) To prove \(\forall n \in \mathbb{Z^+}, F_0 + F_1 + \ldots + F_n = F_{n+2} - 1\) by induction on \(n\):

**Claim:** \(P(0)\)

**Proof:** LHS of \(P(0)\) is \(F_0 = 0\), and
\[
\text{RHS of \(P(0)\) is } F_{0+2} - 1 = F_2 - 1 = 1 - 1 = 0, \text{ so LHS = RHS.}
\]

**Claim:** \(\forall n \in \mathbb{Z^+}, \quad P(n) \implies P(n+1)\)

**Proof:** Let \(n \in \mathbb{Z^+}\) be given,

Suppose \(P(n)\), i.e., \(F_0 + F_1 + \ldots + F_n = F_{n+2} - 1\) \((x)\)
\[
\begin{align*}
(F_0 + F_1 + \ldots + F_n) + F_{n+1} &= (F_{n+2} - 1) + F_{n+1} \quad \text{by \((x)\)} \\
&= (F_{n+2} + F_{n+1}) - 1 \\
&= F_{n+3} - 1 \quad \text{by \((\text{RHS})\)}
\end{align*}
\]

(b) To show that \(\forall n \in \mathbb{Z^+}, \frac{F_0^2 + F_1^2 + \ldots + F_n^2}{F_n F_{n+1}} = F_{n+1} F_{n+2}\) by induction on \(n\):

**Claim:** \(P(0)\)

**Proof:** LHS of \(P(0)\) is \(F_0^2 = 0^2 = 0\), and
\[
\text{RHS of \(P(0)\) is } F_0 F_{0+1} = F_0 F_1 = 0 \cdot 1 = 0, \text{ so LHS = RHS.}
\]

**Claim:** \(\forall n \in \mathbb{Z^+}, \quad P(n) \implies P(n+1)\)

**Proof:** Let \(n \in \mathbb{Z^+}\) be given,

Suppose \(P(n)\), i.e., \(F_0^2 + F_1^2 + \ldots + F_n^2 = F_n F_{n+1}\) \((x)\)
\[
\begin{align*}
(F_0^2 + F_1^2 + \ldots + F_n^2) + F_{n+1}^2 &= F_n F_{n+1} + F_{n+1}^2 \quad \text{by \((x)\)} \\
&= F_{n+1} (F_n + F_{n+1}) = F_{n+1} F_{n+2} \quad \text{by \((\text{RHS})\)}
\end{align*}
\]
(c) To show \( \forall n \in \mathbb{Z}^+ \), \[ F_{2n+1} = F_{n+1}^2 + F_n^2 \land \left[ F_{2n+2} = F_{n+2}^2 - F_n^2 \right] \]

By induction on \( n \):

**Claim:** \( P(0) \)

**Proof:** \( P(0) \iff \left[ F_1 = F_1^2 + F_0^2 \right] \land \left[ F_2 = F_2^2 - F_0^2 \right] \)
\[ \iff \left[ 1 = 1^2 + 0^2 \right] \land \left[ 1 = 1^2 - 0^2 \right] \]
\[ \iff \text{TRUE} \land \text{TRUE} \iff \text{TRUE} \]

**Claim:** \( \forall n \in \mathbb{Z}^+ \), \[ P(n) \implies P(n+1) \]

**Proof:** Let \( n \in \mathbb{Z}^+ \) be given.

Suppose \( P(n) \), i.e., \[ \left[ F_{2n+1} = F_{n+1}^2 + F_n^2 \right] \land \left[ F_{2n+2} = F_{n+2}^2 - F_n^2 \right] \]

Then \( F_{2n+3} = F_{n+2} + F_{n+1} = (F_{n+2}^2 - F_n^2) + (F_{n+1}^2 + F_n^2) = F_{n+2}^2 + F_{n+1}^2 \)

\( \iff (a) \)

\[ (a) \implies (b) \]

AND \( F_{2n+4} = F_{2n+3} + F_{2n+2} = (F_{n+2}^2 + F_{n+1}^2) + (F_{n+2}^2 - F_n^2) \)
\[ \iff (c) \]

\[ (c) \implies (d) \]

\[ (d) = (b) \]

\[ (a) = (b) \]

\[ (c) = (d) \]

\[ \therefore (a) = (b) \land (c) = (d) \]
(d) To show \( \forall n \geq 1, \ F_{n+1}F_{n-1} = F_n^2 + (-1)^n \) by induction on \( n \):

**Claim:** \( P(1) \)

**Proof:** LHS of \( P(1) \) = \( F_2 \cdot F_0 = 1 \cdot 0 = 0 \), and

RHS of \( P(1) \) = \( F_1^2 - 1 = 1^2 - 1 = 0 \), so LHS = RHS.

**Claim:** \( \forall n \geq 1, P(n) \implies P(n+1) \)

**Proof:** Let \( n \geq 1 \) be given

Suppose \( P(n) \), i.e., \( F_{n+1}F_{n-1} = F_n^2 + (-1)^n \) (LHS)

Note \( F_n^2 = F_{n+1}F_{n-1} - (-1)^n = F_{n+1}F_{n-1} + (-1)^{n+1} \) (RHS)

Then \( F_{n+2}F_n = (F_{n+1} + F_n)F_n = F_{n+1}F_n + (F_n^2) \) (LHS)

\[ = F_{n+1}F_n + (F_{n+1}F_{n-1} + (-1)^{n+1}) \text{ by (x)} \]

\[ = F_{n+1}(F_n + F_{n-1}) + (-1)^{n+1} \]

\[ = F_{n+1} \cdot F_{n+1} + (-1)^{n+1} \]

\[ = F_{n+1}^2 + (-1)^{n+1} \] (RHS) \( \Box \)
6. RECALL THE FORMULAE:

\[ F_{2n+1} = F_n^2 + F_{n+1}^2 \]

\[ F_{2n+2} = F_{n+2}^2 - F_n^2 \]

(a) PATTERN-MATCHING,

\[ F_9 = f_5^2 + f_4^2 = 5^2 + 3^2 = 25 + 9 = 34 \]

AND

\[ F_{10} = f_6^2 - f_4^2 = 8^2 - 3^2 = 64 - 9 = 55 \]

(b) AGAIN PATTERN-MATCHING:

\[ F_{16} = f_8^2 + f_7^2 = 21^2 - 13^2 = 441 + 169 = 610 \]

\[ F_{16} = f_9^2 - f_7^2 = 34^2 - 13^2 = 1156 - 169 = 987 \]

AND

\[ F_{17} = f_9^2 + f_8^2 = 1156 + 441 = 1597 \]

(c) ONE MORE TIME:

\[ F_{31} = F_{16}^2 + F_{15}^2 = 987^2 + 610^2 = 1346269 \]

\[ F_{32} = F_{17}^2 - F_{15}^2 = 1597^2 - 610^2 = 2178309 \]

\[ F_{33} = F_{17}^2 + F_{16}^2 = 1597^2 + 987^2 = 3524578 \]

(d) \[ F_{17} = F_{16} + F_{15} \] AND \[ F_{33} = F_{32} + F_{31} \], WHICH YOU CAN SEE IN THE SUMS & DIFFERENCES OF SQUARES!