Basic Logic

Math
Swenton

A ***proposition*** is a well-formed logical statement that is either ***true*** or ***false***, but not both (if it contains *free variables*—more under "Quantifiers" below—it must yield a proposition for every possible assignment of those variables. We can build complicated propositions from simpler ones via logical ***connectives*** and ***quantifiers***, just as we combine numbers by arithmetic operations and, particularly, sets by set operations.

Connectives

take as input some number of propositions and, depending on their values, evaluate to either *true* or *false*. In the same way, each can take some number of predicates, yielding a compound predicate.

not/$\neg$ P

P	T	F
	F	T

P and/$\wedge$ Q

$Q \backslash P$	T	F
T	T	F
F	F	F

P or/$\vee$ Q

$Q \backslash P$	T	F
T	T	T
F	T	F

P implies/$\Rightarrow$ Q

$Q \backslash P$	T	F
T	T	T
F	F	T

P iff/$\Leftrightarrow$ Q

$Q \backslash P$	T	F
T	T	F
F	F	T

P xor/$\oplus$ Q

$Q \backslash P$	T	F
T	F	T
F	T	F

- Note that "or" is *inclusive*—i.e., (*true* or *true*) is *true*—we use "xor" for exclusive-or!
- The connective "$\Rightarrow$" is read "implies"
  - "$P \Rightarrow Q$" means "If P, then Q"; we call P the ***hypothesis*** and Q the ***conclusion*** of the implication.
  - Note that this predicate is not symmetric—e.g., $P \Rightarrow Q$ is not equivalent to $Q \Rightarrow P$.
  - Also, note that *false* $\Rightarrow Q$ for *any* Q—i.e., a false hypothesis allows us to deduce anything we like!
- The predicate "$\Leftrightarrow$" is read "if and only if" (equivalent to "$\Leftarrow$ and $\Rightarrow$"); it serves as our "equal sign" for propositions.

Quantifiers

Quantifiers allow us to specify the roles of variables in a proposition:

- The ***universal quantifier*** $\forall$'s syntax is $\boxed{\forall x,\ P}$, read "for all x, P."
  This is *true* just when P is *true* for each and every value of x; it is *false* if even one choice of x makes P *false*.
- The ***existential quantifier*** $\exists$'s syntax is $\boxed{\exists x \text{ such that } P}$, read "there exists x such that P."
  This is *true* just when there is at least one value of x for which P is *true*; it is *false* if P is *false* for every choice of x.

The values allowed by a quantifier are often restricted implicitly (from context) or explicitly.

  - e.g., "$\forall \varepsilon > 0,\ \exists \delta > 0 \ldots$" refers only to positive real numbers ε and δ.

Unquantified variables in an expression are called ***free variables***; by convention, *free variables are universally quantified*—be aware of free variables and make these quantifiers explicit when necessary (particularly when negating a proposition).

Logical algebra

We can manipulate logical expressions just as we manipulate numerical ones, via the following rules:

Associativity
$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ $\qquad$ $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

Commutativity
$P \wedge Q \Leftrightarrow Q \wedge P$ $\qquad$ $P \vee Q \Leftrightarrow Q \vee P$

Distributivity
$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ $\qquad$ $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Units
$P \wedge \textit{true} \Leftrightarrow P \Leftrightarrow P \vee \textit{false}$ $\qquad$ $P \vee \textit{true} \Leftrightarrow \textit{true}$ $\qquad$ $P \wedge \textit{false} \Leftrightarrow \textit{false}$

Negation
$\neg \textit{true} \Leftrightarrow \textit{false}$ $\qquad$ $\neg(\neg P) \Leftrightarrow P$
$\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$ $\qquad$ $\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$
$\neg(\forall x,\ P) \Leftrightarrow \exists x \text{ such that } \neg P$ $\qquad$ $\neg(\exists x \text{ such that } P) \Leftrightarrow \forall x,\ \neg P$

Implication
$(P \Rightarrow Q) \Leftrightarrow (Q \vee \neg P)$ $\qquad$ $\neg(P \Rightarrow Q) \Leftrightarrow P \wedge \neg Q$
$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$ $\qquad$ $(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
$P \Rightarrow P \vee Q$ $\qquad$ $P \wedge Q \Rightarrow P$ $\qquad$ $\textit{false} \Rightarrow P$ $\qquad$ $P \Rightarrow \textit{true}$

Other identities
$\textit{false} \wedge P \Leftrightarrow \textit{false}$ $\qquad$ $P \wedge P \Leftrightarrow P$ $\qquad$ $P \wedge (\neg P) \Leftrightarrow \textit{false}$
$\textit{true} \vee P \Leftrightarrow \textit{true}$ $\qquad$ $P \vee P \Leftrightarrow P$ $\qquad$ $P \vee (\neg P) \Leftrightarrow \textit{true}$

Sets

A ***set*** is a collection of objects, such that any object x is either in the set (written $x \in S$) or not in the set (written $x \notin S$), but not both: $\boxed{S \text{ is a set } \Leftrightarrow \forall x,\ x \in S \oplus x \notin S}$.

- Simple sets can be expressed simply by listing their elements — e.g., "the set $\{a, b, c\}$", or "the set $A = \{1, 2, 3, \ldots\}$".
- More complicated sets are often expressed via the notation $\{x : P(x)\}$, read "the set of all x such that $P(x)$."
  This allows us to collect all objects with some property into a set: $\boxed{a \in \{x : P(x)\} \text{ means "}P(a) \text{ is } \textit{true}\text{"}}$.
  · e.g., $a \in \{x : x^2 - 3x + 2 = 0\}$ simply means $a^2 - 3a + 2 = 0$.
- Some sets are expressed via the more intricate notation $\{f(x) : P(x)\}$, in which the left side isn't simply a variable.
  Here, the left side indicates the *form* of the set's elements, and the right side indicates the *credentials* required for inclusion into the set; in practical terms, $\boxed{a \in \{f(x) : P(x)\} \text{ means "}a = f(x), \text{ where } P(x) \text{ is } \textit{true}\text{"}}$.
  · e.g., $z \in \{x + iy : x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}$ means: $z = x + iy$, where $x, y \in \mathbb{R}$ and $x^2 + y^2 = 1$.
- Some common sets and the symbols used to represent them:
  - The ***empty set***: $\varnothing = \{\ \}$, i.e., the set containing no elements
  - Some important sets of numbers:
    · the ***natural numbers***, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and ***integers***, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$;
    · the ***rationals***, $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$ and ***real numbers***, $\mathbb{R}$; and
    · the ***complex numbers***, $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ (where $i^2 = -1$).

Set Arithmetic

Numbers and logical propositions are not the only objects that can be manipulated and compared—similar operations exist for sets, which form the foundation of most objects that we in mathematics; the most important of these are listed below.
[In the formulæ below, uppercase letters represent sets and script letters represent collections of sets.]

Subsets	$A \subset B$ means $x \in A \Rightarrow x \in B$	
Equality	$A = B$ means $x \in A \Leftrightarrow x \in B$	[or, equivalently, $A \subset B \wedge B \subset A$]
Union	$A \cup B \overset{\text{def}}{=} \{x : x \in A \vee x \in B\}$	$\bigcup \mathscr{B} \overset{\text{def}}{=} \{x : \exists B \in \mathscr{B} \text{ such that } x \in B\}$
Intersection	$A \cap B \overset{\text{def}}{=} \{x : x \in A \wedge x \in B\}$	$\bigcap \mathscr{B} \overset{\text{def}}{=} \{x : \forall B \in \mathscr{B},\ x \in B\}$
Difference	$A \smallsetminus B \overset{\text{def}}{=} \{x : x \in A \wedge x \notin B\}$	$A \triangle B \overset{\text{def}}{=} \{x : x \in A \oplus x \in B\}$
Cartesian product	$A \times B \overset{\text{def}}{=} \{(a, b) : a \in A \wedge b \in B\}$	
Power set	$\mathscr{P}(X) \overset{\text{def}}{=} \{A : A \subset X\}$	[so $s \in \mathscr{P}(X) \Leftrightarrow s \subset X$]
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cap \bigcup_{B \in \mathscr{B}} B = \bigcup_{B \in \mathscr{B}} (A \cap B)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cup \bigcap_{B \in \mathscr{B}} B = \bigcap_{B \in \mathscr{B}} (A \cup B)$
DeMorgan's laws	$X \smallsetminus (A \cup B) = (X \smallsetminus A) \cap (X \smallsetminus B)$	$X \smallsetminus \bigcup_{B \in \mathscr{B}} B = \bigcap_{B \in \mathscr{B}} (X \smallsetminus B)$
	$X \smallsetminus (A \cap B) = (X \smallsetminus A) \cup (X \smallsetminus B)$	$X \smallsetminus \bigcap_{B \in \mathscr{B}} B = \bigcup_{B \in \mathscr{B}} (X \smallsetminus B)$
Emptiness	$A \neq \varnothing \Leftrightarrow \exists a \in A$	
	A and B are called ***disjoint*** if $A \cap B = \varnothing$	
	$\mathscr{C}$ is called a ***[pairwise] disjoint collection*** if $A, B \in \mathscr{C} \Rightarrow A = B$ or $A \cap B = \varnothing$	