Courtesy Department of Computer Science,
University of Western Ontario

Master Theorem:
Practice Problems and Solutions

Master Theorem

The Master Theorem applies to recurrences of the following form:

① IDENTIFY a, b, & f(n) AND CHECK BASE CONDITIONS

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

② FIND THE CRITICAL EXPONENT, $\log_b a = \frac{\log a}{\log b}$

③

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with[1] $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.
   Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

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Practice Problems

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

COMPARE f(n) TO $n^{\text{CRITICAL EXPONENT}}$:

1. f(n) IS O(n TO A POWER LESS THAN THE CRITICAL EXP.)
2. f(n) IS Θ(n TO EXACTLY THE CRITICAL EXP. ... POSSIBLY WITH SOME log n's)
3. f(n) IS Ω(n TO A POWER GREATER THAN THE CRITICAL EXP.)

* NEED TO CHECK REGULARITY FOR 3.

IF ALL WORKS, THE THEOREM TELLS YOU HOW FAST T(n) GROWS!

1. $T(n) = 3T(n/2) + n^2$
2. $T(n) = 4T(n/2) + n^2$
3. $T(n) = T(n/2) + 2^n$
4. $T(n) = 2^n T(n/2) + n^n$
5. $T(n) = 16T(n/4) + n$
6. $T(n) = 2T(n/2) + n \log n$

[1]most of the time, $k = 0$