Function Basics

Functions

- Suppose that X and Y are sets; a ***function*** $f : X \to Y$ is a relation from X to Y, which we could write as $x \overset{f}{\mapsto} y$ or $f : x \mapsto y$, having two properties:
  - $\forall\, x \in X,\ \exists\, y \in Y$ with $x \overset{f}{\mapsto} y$ [every $x \in X$ has some corresponding $y \in Y$], and
  - $(x \overset{f}{\mapsto} y) \wedge (x \overset{f}{\mapsto} y') \ \Rightarrow\ y = y'$ [each $x \in X$ corresponds to just one $y \in Y$].

  For each $x \in X$, we denote this unique $y \in Y$ with $x \overset{f}{\mapsto} y$ by $f(x)$.
- Independent of its formal definition as a relation, the best way to think about a function $f : X \to Y$ is as an active operation that picks up each element of X and ***maps*** it (sends it) to some element of Y:

  A function $f : X \to Y$ is a rule that assigns to each $x \in X$ exactly one value $f(x) \in Y$.
- The second property in the formal definition of function could be loosely written as "$x = x' \ \Rightarrow\ f(x) = f(x')$"; in words, this says a function must be ***well-defined***, i.e., if you give it the same inputs, it produces the same outputs. We use this property all the time with little note—e.g., every time that we apply a function to both sides of an equation.
- We often express functions via expressions, as in "the function $f(x) = x^2 + 1$".
  - This should *not* be viewed as an equation, but rather, as a convenient way of expressing the *rule* $f : x \mapsto x^2 + 1$. In this case, the domain and codomain are often clear from context; however, it is always best to explicitly state the domain, codomain, and rule for a function, i.e.: "the function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2 + 1$."
- Suppose that $f : X \to Y$ is a function; then we define the following terms:
  - The ***domain*** of f is X. [its set of input values]
  - The ***codomain*** of f is Y. [the set into which it sends its results—note that it need not fill that set!]
  - Two functions $f, f' : X \to Y$ are ***equal*** $(f = f')$ when $\forall\, x \in X, f(x) = f'(x)$ [they represent the same rule]
  - The ***range*** (or *image*) of f is the set $\{\, f(x) \ :\ x \in X \,\} \subset Y$ [the subset of Y that actually gets hit by f]
    - Note that this could be more explicity expressed as $\{\, y \in Y \ :\ \exists\, x \in X \text{ with } f(x) = y \,\}$.
  - f is ***injective*** means $f(x) = f(x') \ \Rightarrow\ x = x'$. [equivalently, that $x \neq x' \ \Rightarrow\ f(x) \neq f(x')$]
    - This means that no two distinct elements of x map to the same element of Y (a.k.a. f is "one-to-one").
    - Note well that this is the *converse* of the second part of what it means to be a function: we can always apply a function to both sides of an equation; for an injective function, we can also *remove* it from both sides!
  - f is ***surjective*** means that the range of f is all of Y. [equivalently, $\forall\, y \in Y,\ \exists\, x \in X$ with $f(x) = y$]
    - This means that every element of Y is "hit" by some element of X (a.k.a. f is "onto").
  - f is ***bijective*** means that f is both injective and surjective.
    - A bijective function $f : X \to Y$ *pairs* each element of X with exactly one element of Y, and vice-versa.

Compositions and inverses

- The ***composition*** of two functions $f : X \to Y$ and $g : Y \to Z$ simply applies one and then the other:

  $$g \circ f : X \to Z \text{ is given by } (g \circ f)(x) = g\big(f(x)\big).$$

  [Note that function compositions are read right-to-left, because the inputs are written on the right!]
- If X is a set, the ***identity*** function $\mathrm{id}_X : X \to X$ does nothing: $\mathrm{id}_X : x \mapsto x$.
  - It is very quick to prove that this function is bijective!
- Two functions $f : X \to Y$ and $g : Y \to X$ are ***inverses*** means that they "undo" each other:

  $$\forall\, x \in X,\ g\big(f(x)\big) = x \ \ \wedge\ \ \forall\, y \in Y,\ f\big(g(y)\big) = y.$$

  - Intuitively, f and g send elements back where the other function mapped them from, "reversing" each others' arrows.
  - In terms of compositions, this can be very cleanly stated as $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.
  - $f : X \to Y$ has an inverse just when f is bijective, and in this case, its inverse is unique; we denote it by f^{-1}.

    [This is an inverse, not a power!]